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On Some New Nonlinear Retarded Integral Inequalities with Iterated Integrals and their Applications in Integro-Differential Equations

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Abstract

In this paper, some new generalized retarded nonlinear integral inequalities of Gronwall-Bellman type are discussed. The upper bounds estimation of the embedded unknown functions are discussed by integral and differential techniques. Our results generalize some inequalities of H. El-Owaidy et al. [1] with both retard and nonlinear integral. Some applications are also presented in order to illustrate the usefulness of some of our results.

Keywords: Retarded integral inequalities; estimation; iterated integrals; analysis technique; integro-differential equations.

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1 Introduction

Integral inequalities that give explicit bounds on unknown functions provide a very useful and important device in the study of many qualitative as well as quantitative properties of solutions of nonlinear differential equations and integral equations. In 1919, Gronwall [2] introduced the famous Gronwall inequality (Theorem 1.1) while investigating the dependence of systems of differential equations with respect to a parameter. Integral inequalities of Gronwall type are important tools in the study of existence, uniqueness, boundedness, stability, invariant manifolds and other qualitative properties of solutions of differential equations. Since then, a lot of contributions have been achieved by many researchers. The original Gronwall inequality has been extended to the more general cases including the generalized linear and nonlinear Gronwall type inequalities and mixed Gronwall-Bellman inequalities (see [1], [3-12]).

Theorem 1.1 (Gronwall [2]). Let $u(t)$ be a continuous function defined on the interval $D = [\alpha, \alpha + h]$ and

$$0 \leq u(t) \leq \int_{\alpha}^{t} [bu(s) + a]ds, \forall t \in D,$$

where α, a, b and h are nonnegative constants. Then

$$0 \leq u(t) \leq ahe^{bh}, \forall t \in D.$$

Recently, H. El-Owaidy et al. [1] studied the following nonlinear integral inequalities of Gronwall type

Theorem 1.2 (H. El-Owaidy et al. [1]). Let $u(t)$ be a real-valued positive continuous function and $f(t), g(t)$ are real-valued non-negative continuous functions defined on $I = [0, \infty)$ and satisfy the inequality

$$u^p(t) \leq u_0 + \int_0^t f(s)\left[u^p(s) + \int_0^s g(\lambda)u(\lambda)d\lambda\right]ds, \forall t \in I,$$

where u_0 and p are positive constant. Then

$$u(t) \leq u_0^{\frac{1}{p}}\left[1 + \int_0^t f(s)\exp\left(\int_0^s [f(\lambda) + g(\lambda)]d\lambda\right)ds\right]^{\frac{1}{p}}, \qquad \forall t \in I.$$

Theorem 1.3 (H. El-Owaidy et al. [1]). Let $u(t)$ be a real-valued positive continuous function and $f(t), g(t)$ are real-valued nonnegative continuous functions defined on $I = [0, \infty)$ and satisfy the inequality

$$u^p(t) \leq u_0 + \int_0^t f(s)\left[u^q(s) + \int_0^s g(\lambda)u(\lambda)d\lambda\right]ds, \forall t \in I,$$

where u_0, p and q are constants such that $u_0 \geq 0$ and $p > q \geq 0$. Then

$$u(t) \leq \left[u_0 + \int_0^t f(s)K(s)\exp\left(\int_0^s g(\lambda)d\lambda\right)ds\right]^{\frac{1}{p}}, \qquad \forall t \in I,$$

where

$$K(t) = \left[u_0^{\frac{q[p-q]}{p}} + \left[\frac{q[p-q]}{p}\right]\int_0^t f(s)\exp\left(-[p-q]\int_0^s g(\lambda)d\lambda\right)ds\right]^{[\frac{1}{p-q}]}, \forall t \in I.$$

Theorem 1.4 (H. El-Owaidy et al. [1]). Let $u(t)$ be a real-valued positive continuous function and $f(t), g(t)$ are nonnegative real-valued continuous functions defined on $I = [0, \infty)$, and satisfy the inequality

$$u(t) \leq u_0 + \int_0^t f(s)u(s)\left[u^q(s) + \int_0^s g(\lambda)u(\lambda)d\lambda\right]^p ds, \forall t \in I,$$

where u_0, p and q, are positive constants such that $p+q>1$, . Then

$$u(t) \leq u_0 \exp\bigg(\int_0^t f(s)K_1(s)ds\bigg), \qquad t \in I,$$

where

$$K_1(t) = \frac{u_0^{pq}\exp\bigg(p\int_0^t g(s)ds\bigg)}{\bigg[1-q[p+q-1]u_0^{q[p+q-1]}\int_0^t f(s)\exp([p+q-1]\int_0^s g(\lambda)d\lambda)ds\bigg]^{[\frac{p}{p+q-1}]}},$$

such that, $q[p+q-1]u_0^{q[p+q-1]}\int_0^t f(s)\exp([p+q-1]\int_0^s g(\lambda)d\lambda)ds < 1, \forall t \in I.$

Theorem 1.5 (H. El-Owaidy et al. [1]). Let $u(t)$ be a real-valued positive continuous function and $f(t), g(t)$ are nonnegative real-valued continuous functions defined on $I=[0,\infty)$, and $n(t)$ be a positive monotonic nondecreasing continuous function on $I=[0,\infty)$ and satisfy the inequality

$$u(t) \leq n(t) + \int_0^t f(s)\bigg[u(s) + \int_0^s g(\lambda)u(\lambda)d\lambda\bigg]^p ds, \forall t \in I,$$

where p be a constant such that $p \in (0,1)$. Then

$$u(t) \leq n(t)\bigg[1 + \int_0^t f(s)K_2(s)n^{[p-1]}(s)\bigg], \qquad \forall t \in I,$$

where

$$K_2(t) = \exp\bigg(p(1-p)\int_0^t g(s)ds\bigg)\bigg[1+(1-p)\int_0^t f(s)n^{[p-1]}(s) \times \exp\bigg(-(1-p)\int_0^s g(\lambda)d\lambda\bigg)ds\bigg]^{[\frac{p}{1-p}]}, \quad \forall t \in I.$$

However, many real life problems that have in the past, sometimes been modeled by initial value problems for differential equations. Actually involve a significant memory effect that can be represented in a more refined model using a differential equation incorporating retarded or delayed arguments. In this situations, we need to discuss some retarded nonlinear integral inequalities (see [13-19]). So, the given bound on such inequalities in [1] are not directly applicable in the study of certain retarded nonlinear differential and integral equations. It is desirable to establish new inequalities of the above type, which can be used more effectively in the study of certain classes of retarded nonlinear differential, integral and integro-differential equations.

In this paper, we extend certain results that where proved in [1] to obtain new generalizations of formerly famous Gronwall type inequalities where non retarded case t in [1] is changed into retarded case $\alpha(t)$, and give upper bound estimation of the unknown function by integral and differential techniques in order to study retarded differential, integral and integro-differential equations.

2 Main Results

In this section, several new retarded integral inequalities of Gronwall-Bellman type are introduced.

Throughout this article, $\mathbb{R}$ denoted the set of real numbers; $I=[0,\infty)$, $\mathbb{R}_+=(0,\infty)$, $\mathbb{R}_1=[1,\infty)$ and $'$ denotes the derivative. $\mathcal{C}(I,I)$ denotes the set of all nonnegative real-valued continuous functions from I into I and $\mathcal{C}^1(I,I)$ denotes the set of all nonnegative real-valued continuously differentiable functions from I into I.

Theorem 2.1. *Let $u(t)$, $g(t)$, $f(t) \in \mathcal{C}(I, I), \alpha(t) \in \mathcal{C}^1(I, I)$ be nondecreasing with $\alpha(t) \leq t$ on I and satisfy the inequality*

$$u^p(t) \leq u_0 + \int_0^{\alpha(t)} f(s)\Big[u^p(s) + \int_0^s g(\lambda)u(\lambda)d\lambda\Big]ds, \qquad \forall t \in I, \tag{2.1}$$

where u_0 and p are positive constants. Then

$$u(t) \leq u_0^{\frac{1}{p}}\left[1 + \int_0^{\alpha(t)} f(s)\exp\left(\int_0^s [f(\lambda) + g(\lambda)]d\lambda\right)ds\right]^{\frac{1}{p}}, \qquad \forall t \in I. \tag{2.2}$$

Proof. Let $J^p(t)$ equal the right hand side in (2.1), we have $J(0) = u_0^{\frac{1}{p}}$ and

$$u(t) \leq J(t), u(\alpha(t)) \leq J(\alpha(t)) \leq J(t), \qquad \forall t \in I. \tag{2.3}$$

Differentiating $J^p(t)$ with respect to t, and using (2.3) leads to

$$pJ^{[p-1]}(t)\frac{dJ(t)}{dt} \leq \alpha'(t)f(\alpha(t))Y(t), \qquad \forall t \in I, \tag{2.4}$$

where $Y(t) = J^p(t) + \int_0^{\alpha(t)} g(s)J(s)ds$, thus we have $Y(0) = J^p(0) = u_0$ and

$$J(t) \leq Y(t), \qquad \forall t \in I. \tag{2.5}$$

Differentiating $Y(t)$ with respect to t, and using (2.4),(2.5) leads to

$$\frac{dY(t)}{dt} \leq \alpha'(t)[f(t) + g(t)]Y(t), \qquad \forall t \in I.$$

By taking $t = s$ in the above inequality and integrating from 0 to t, we get

$$Y(t) \leq u_0\exp\left(\int_0^{\alpha(t)} [f(s) + g(s)]ds\right), \qquad \forall t \in I, \tag{2.6}$$

from (2.6) and (2.4), we obtain

$$J^{[p-1]}(t)\frac{dJ(t)}{dt} \leq \frac{1}{p}u_0\alpha'(t)f(\alpha(t))\exp\left(\int_0^{\alpha(t)} [f(s) + g(s)]ds\right), \qquad \forall t \in I.$$

The above inequality implies an estimation of $J(t)$ as follows

$$J(t) \leq u_0^{\frac{1}{p}}\left[1 + \int_0^{\alpha(t)} f(s)\exp\left(\int_0^s [f(\lambda) + g(\lambda)]d\lambda\right)ds\right]^{\frac{1}{p}}, \qquad \forall t \in I. \tag{2.7}$$

Using (2.7) in (2.3), we get the required inequality in (2.2). The proof is completed. □

Remark 2.1. If $\alpha(t) = t$, then Theorem 2.1 reduces to Theorem 1.2.

Theorem 2.2. *Let $u(t), g(t), f(t) \in \mathcal{C}(I, I), \alpha(t) \in \mathcal{C}^1(I, I)$ be nondecreasing with $\alpha(t) \leq t, \alpha(0) = 0$ on I and satisfy the inequality*

$$u^p(t) \leq u_0 + \int_0^{\alpha(t)} f(s)\Big[u^q(s) + \int_0^s g(\lambda)u(\lambda)d\lambda\Big]ds, \qquad \forall t \in I, \tag{2.8}$$

where $u_0 > 0, p > q \geq 1$ are constants. Then

$$u(t) \leq \left[u_0 + \int_0^{\alpha(t)} f(s)k(\alpha^{-1}(s))ds\right]^{\frac{1}{p}}, \qquad \forall t \in I, \tag{2.9}$$

where

$$k(t) = \exp\left(\int_0^{\alpha(t)} g(s)ds\right)\left[u_0^{\frac{q[p-q]}{p}} + \left[\frac{q[p-q]}{p}\right]\int_0^{\alpha(t)} f(s)\exp\left(-(p-q)\int_0^s g(\lambda)d\lambda\right)ds\right]^{[\frac{1}{p-q}]}. \tag{2.10}$$

Proof. Let $J_1^p(t)$ equal the right hand side in (2.8), we have $J_1(0) = u_0^{\frac{1}{p}}$ and

$$u(t) \leq J_1(t), u(\alpha(t)) \leq J_1(\alpha(t)) \leq J_1(t), \qquad \forall t \in I. \tag{2.11}$$

Differentiating $J_1^p(t)$, with respect to t, using (2.11) we get

$$pJ_1^{[p-1]}(t)\frac{dJ_1(t)}{dt} \leq \alpha'(t)f(\alpha(t))Y_1(t), \qquad \forall t \in I, \tag{2.12}$$

where $Y_1(t) = J_1^q(t) + \int_0^{\alpha(t)} g(s)J_1(s)ds$, thus we have $Y_1(0) = J_1^q(0) = u_0^{\frac{q}{p}}$, and $J_1^q(t) \leq Y_1(t)$, but $q \geq 1$, thus

$$J_1(t) \leq Y_1(t), \qquad \forall t \in I. \tag{2.13}$$

Differentiating $Y_1(t)$ with respect to t, and using (2.12), (2.13) leads to

$$\frac{dY_1(t)}{dt} \leq \frac{q}{p}\alpha'(t)f(\alpha(t)Y_1^{[q-p+1]}(t) + \alpha'(t)g(\alpha(t))Y_1(t), \qquad \forall t \in I,$$

but $Y_1(t) > 0$, thus we have

$$Y_1^{[p-q-1]}(t)\frac{dY_1(t)}{dt} - \alpha'(t)g(\alpha(t))Y_1^{[p-q]}(t) \leq \frac{q}{p}\alpha'(t)f(\alpha(t)), \qquad \forall t \in I. \tag{2.14}$$

Now, if we put

$$Z(t) = Y_1^{[p-q]}(t), \qquad \forall t \in I, \tag{2.15}$$

then we have $Z(0) = Y_1^{[p-q]}(0) = u_0^{\frac{q[p-q]}{p}}$ and $Y_1^{[p-q-1]}(t)\frac{dY_1(t)}{dt} = [\frac{1}{p-q}]\frac{dZ}{dt}$, thus from (2.14) we obtain

$$\frac{dZ(t)}{dt} - [p-q]\alpha'(t)g(\alpha(t))Z(t) \leq [\frac{q[p-q]}{p}]\alpha'(t)f(\alpha(t)), \qquad \forall t \in I. \tag{2.16}$$

The above inequality implies the following estimations of $Z(t)$, such that

$$Z(t) \leq k^{[p-q]}(t), \qquad \forall t \in I,$$

where $k(t)$ is as defined in (2.10). From (2.8) and the above inequality we have

$$Y_1(t) \leq k(t), \qquad \forall t \in I,$$

thus from(2.12) and the above inequality we have

$$pJ_1^{[p-1]}(t)\frac{dJ_1(t)}{dt} \leq \alpha'(t)f(\alpha(t))k(t), \qquad \forall t \in I.$$

By taking $t = s$ in the above inequality and integrating from 0 to t, gives

$$J_1(t) \leq \left[u_0 + \int_0^{\alpha(t)} f(s)k(\alpha^{-1}(s))\right]^{\frac{1}{p}}, \qquad \forall t \in I. \tag{2.17}$$

Using (2.17) in (2.11) leads to the required inequality in (2.9). The proof is completed. □

Remark 2.2. If $\alpha(t) = t$, then Theorem 2.2 reduces to Theorem 1.3.

Theorem 2.3. *Let* $u(t), g(t), f(t) \in \mathcal{C}(I, I), \alpha(t) \in \mathcal{C}^1(I, I)$ *be nondecreasing with* $\alpha(t) \le t, \alpha(0) = 0,$ *on* I *and satisfy the inequality*

$$u(t) \le u_0 + \int_0^{\alpha(t)} f(s)u(s)\left[u^q(s) + \int_0^s g(\lambda)u(\lambda)d\lambda\right]^p ds, \forall t \in I, \tag{2.18}$$

where u_0, p *and* q *are positive constants such that* $p + q > 1$. *Then*

$$u(t) \le u_0 \exp\bigg(\int_0^{\alpha(t)} f(s)k_1(\alpha^{-1}(s))ds\bigg), \qquad \forall t \in I, \tag{2.19}$$

where

$$k_1(t) = \frac{u_0^{pq} \exp\bigg(p \int_0^{\alpha(t)} g(s)ds\bigg)}{\bigg[1 - q[p+q-1]u_0^{q[p+q-1]} \int_0^{\alpha(t)} f(s) \exp([p+q-1] \int_0^s g(\lambda)d\lambda)ds\bigg]^{[\frac{p}{p+q-1}]}}, \tag{2.20}$$

such that, $q[p+q-1]u_0^{q[p+q-1]} \int_0^{\alpha(t)} f(s) \exp([p+q-1] \int_0^s g(\lambda)d\lambda)ds < 1, \forall t \in I.$

Proof. Let $J_2(t)$ equal the right hand side of (2.18), which is a positive and nondecreasing function on I with $J_2(0) = u_o$. Then (2.18) reduces to

$$u(t) \le J_2(t), u(\alpha(t)) \le J_2(\alpha(t)) \le J_2(t), \qquad \forall t \in I. \tag{2.21}$$

Differentiating $J_2^p(t)$, with respect to t, using (2.21) we have

$$\frac{dJ_2(t)}{dt} \le \alpha'(t)f(\alpha(t))J_2(t)Y_2^p(t), \qquad \forall t \in I, \tag{2.22}$$

where $Y_2(t) = J_2^q(t) + \int_0^{\alpha(t)} g(s)J_2(s)ds$, hence $Y_2(0) = J_2^q(0) = u_0^q$, and

$$J_2(t) \le Y_2(t) \qquad \forall t \in I. \tag{2.23}$$

Differentiating $Y_2(t)$ with respect to t, and using (2.22), (2.23), leads to

$$\frac{dY_2(t)}{dt} \le q\alpha'(t)f(\alpha(t))Y_2^{[p+q]}(t) + \alpha'(t)g(\alpha(t))Y_2(t), \qquad \forall t \in I,$$

but $Y_2(t) > 0$, then we have

$$Y_2^{-[p+q]}(t)\frac{dY_2(t)}{dt} - \alpha'(t)g(\alpha(t))Y_2^{[1-(p+q)]}(t) \le q\alpha'(t)f(\alpha(t)), \qquad \forall t \in I. \tag{2.24}$$

If we let

$$Y_2^{[1-(p+q)]}(t) = Z_1(t), \qquad \forall t \in I, \tag{2.25}$$

we have $Z_1(0) = Y_2^{-[p+q-1]}(0) = u_0^{-q[p+q-1]}$, and $Y_2^{-[p+q]}(t)\frac{dY_2(t)}{dt} = \frac{-1}{[p+q-1]}\frac{dZ_1}{dt}$, then we can write the inequality (2.24) as follows

$$\frac{dZ_1}{dt} + (p+q-1)\alpha'(t)g(\alpha(t))Z_1(t) \ge -q(p+q-1)\alpha'(t)f(\alpha(t)), \qquad \forall t \in I. \tag{2.26}$$

The above inequality implies the following estimation of $Z_1(t)$ such that

$$Z_1(t) \ge [k_1(t)]^{\frac{-[p+q-1]}{p}}, \qquad \forall t \in I,$$

where $k_1(t)$ as defined in (2.20). From (2.25) and the above inequality, we have

$$Y_2^p(t) \le k_1(t), \qquad \forall t \in I,$$

thus from the above inequality in (2.22), we have

$$\frac{dJ_2(t)}{dt} \le \alpha'(t) f(\alpha(t)) k_1(t), \qquad \forall t \in I.$$

By taking $t = s$ in the above inequality and integrating from 0 to t , produces

$$J_2(t) \le u_0 \exp\bigg(\int_0^{\alpha(t)} f(s) k_1(\alpha^{-1}(s)) ds \bigg), \qquad t \in I. \tag{2.27}$$

Using (2.27) in (2.21), we get the required inequality in (2.19). The proof is completed. □

Remark 2.3. If $q = 1$, then Theorem 2.3 reduces to Theorem 3 in [16].

Remark 2.4. If $\alpha(t) = t$, then Theorem 2.3 reduces to Theorem 1.4.

Remark 2.5. If $q = 1$ and $\alpha(t) = t$, then Theorem 2.3 reduces to Theorem 3.2 in [3].

Theorem 2.4. *Let $u(t), g(t), f(t) \in \mathcal{C}(I, I), \alpha(t) \in \mathcal{C}^1(I, I)$ be nondecreasing with $\alpha(t) \le t, \alpha(0) = 0,$ on $I, n(t) \in \mathcal{C}(I, \mathbb{R}_+)$ be nondecreasing and satisfy the inequality*

$$u(t) \le n(t) + \int_0^{\alpha(t)} f(s) \bigg[u(s) + \int_0^s g(\lambda) u(\lambda) d\lambda \bigg]^p ds, , \quad \forall t \in I, \tag{2.28}$$

where p be a constant such that $p \in (0, 1)$. Then

$$u(t) \le n(t) \bigg[1 + \int_0^{\alpha(t)} f(s) n^{[p-1]}(s) k_2(\alpha^{-1}(s)) \bigg], \qquad \forall t \in I, \tag{2.29}$$

where

$$\begin{aligned} k_2(t) = & \exp\bigg(p(1-p) \int_0^{\alpha(t)} g(s) ds \bigg) \bigg[1 + (1-p) \int_0^{\alpha(t)} f(s) n^{[p-1]}(s) \\ & \times \exp\bigg(-(1-p) \int_0^s g(\lambda) d\lambda \bigg) ds \bigg]^{[\frac{p}{1-p}]}, \quad \forall t \in I. \end{aligned} \tag{2.30}$$

Proof. Since $n(t)$ is a positive monotonic nondecreasing function, we observe from the inequality (2.28)

$$\bigg[\frac{u(t)}{n(t)} \bigg] \le 1 + \int_0^{\alpha(t)} f(s) n^{[p-1]}(s) \bigg[\bigg(\frac{u(s)}{n(s)} \bigg) + \int_0^{\alpha(t)} g(\lambda) \bigg(\frac{u(\lambda)}{n(\lambda)} \bigg) d\lambda \bigg]^p ds, \quad \forall t \in I.$$

Let

$$m(t) = \frac{u(t)}{n(t)}, \qquad m(0) \le 1, \qquad \forall t \in I. \tag{2.31}$$

Hence

$$m(t) \le 1 + \int_0^{\alpha(t)} f(s) n^{[p-1]}(s) \bigg[m(s) + \int_0^t g(\lambda) m(\lambda) d(\lambda) \bigg]^p, \qquad \forall t \in I. \tag{2.32}$$

Let $J_3(t)$ equal the right hand side of (2.32), which is a positive and nondecreasing function on I with $J_3(0) = 1$. Then (2.32) reduces to

$$m(t) \le J_3(t), m(\alpha(t) \le J_3(\alpha(t)) \le J_3(t) \qquad \forall t \in I. \tag{2.33}$$

Differentiating $J_3(t)$ with respect to t, using (2.33), we get

$$\frac{dJ_3(t)}{dt} \leq \alpha'(t)f(\alpha(t))n^{[p-1]}(\alpha(t))Y_3^p(t), \qquad \forall t \in I, \tag{2.34}$$

where $Y_3(t) = J_3(t) + \int_0^{\alpha(t)} g(s)J_3(s)d(s)$, $Y_3(0) = J_3(0) = 1$, and

$$J_3(t) \leq Y_3(t), \qquad \forall t \in I. \tag{2.35}$$

Differentiating $Y_3(t)$ with respect to t and using (2.34), (2.35) we get

$$\begin{aligned} \frac{dY_3(t)}{dt} &= \frac{dJ_3(t)}{dt} + \alpha'(t)g(\alpha(t))J_3(t) \\ &\leq \alpha'(t)f(\alpha(t))n^{[p-1]}(\alpha(t))Y_3^p(t) + \alpha'(t)g(\alpha(t))Y_3(t), \forall t \in I. \end{aligned}$$

The above inequality implies the following estimation of $Y_3(t)$, such that

$$Y_3^p(t) \leq k_2(t), \qquad \forall t \in I,$$

where $k_2(t)$ is as given in (2.30). From the above inequality in (2.34), we have

$$\frac{dJ_3(t)}{dt} \leq \alpha'(t)f(\alpha(t))n^{[p-1]}(t)k_2(t), \qquad \forall t \in I,$$

the above inequality implies the following estimation of $J_3(t)$ such that

$$J_3(t) \leq 1 + \int_0^{\alpha(t)} f(s)k_2(\alpha^{-1}(s))n^{[p-1]}ds, \qquad \forall t \in I.$$

Using the above inequality in (2.33), we get

$$m(t) \leq 1 + \int_0^{\alpha(t)} f(s)k_2(\alpha^{-1}(t))n^{[p-1]}ds, \qquad \forall t \in I. \tag{2.36}$$

We get the desired bound in (2.29) from (2.31) and (2.36). The proof is completed. □

Remark 2.6. If $\alpha(t) = t$, then Theorem 2.4 reduces to Theorem 1.5.

Theorem 2.5. *Let $u(t), g(t), f(t) \in \mathcal{C}(I, I)$ be nondecreasing, we assume that $\psi_1(t), \psi_2(t)$ are nondecreasing and continuous functions defined on I with $\psi_i(t) > 0, \forall t > 0, i = 1, 2$, and $\alpha(t) \in \mathcal{C}^1(I, I)$ is a nondecreasing function with $\alpha(t) \leq t, \alpha(0) = 0$ and satisfy the inequality*

$$u(t) \leq u_0 + \int_0^{\alpha(t)} f(s)\psi_1(u(s))\left[u^q(s) + \int_0^s g(\lambda)\psi_2(u(\lambda))d\lambda\right]^p ds, \forall t \in I, \tag{2.37}$$

where $u_0 > 0, p > 0$ and $q \geq 1$, are constants such that $p + q > 1$. Then

$$u(t) \leq \Psi_1^{-1}\left[\Psi_2^{-1}\left(\Psi_2\left(\Psi_1(u_0^q) + \int_0^{\alpha(t)} g(s)ds\right) + \int_0^{\alpha(t)} f(s)ds\right)\right], \tag{2.38}$$

for all $t \in [0, T_1]$, where

$$\Psi_1(r) = \int_1^r \frac{dt}{\psi_2(t)}, r > 0, \tag{2.39}$$

$$\Psi_2(r) = \int_1^r \frac{\psi_2(\Psi_1^{-1}(s))ds}{\psi_1(\Psi_1^{-1}(s))(\Psi_1^{-1}(s))^{p+q-1}}, r > 0, \tag{2.40}$$

and T_1 is the largest number such that

$$\Psi_2\left(\Psi_1(u_0^q) + \int_0^{\alpha(t)} g(s)ds\right) + \int_0^{\alpha(t)} f(s)ds \leq \frac{\psi_2(\Psi_1^{-1}(s))ds}{\psi_1(\Psi_1^{-1}(s))(\Psi_1^{-1}(s))^{p+q-1}},$$

$$\Psi_2^{-1}\left(\Psi_2\left(\Psi_1(u_0^q) + \int_0^{\alpha(t)} g(s)ds\right) + \int_0^{\alpha(t)} f(s)ds\right) \leq \frac{dt}{\psi_2(t)}.$$

Proof. Let $J_4(t)$ equal the right hand side in (2.37), which is a positive and nondecreasing function on I, we have

$$u(t) \leq J_4(t), u(\alpha(t)) \leq J_4(\alpha(t)) \leq J_4(t), J_4(0) = u_0, \forall t \in I. \tag{2.41}$$

Differentiating $J_4(t)$ with respect to t and using (2.41), we get

$$\frac{dJ_4(t)}{dt} \leq \alpha'(t) f(\alpha(t)) \psi_1(J_4(t)) Y_4^p(t), \qquad \forall t \in I, \tag{2.42}$$

where $Y_4(t) = J_4^q(t) + \int_0^{\alpha(t)} g(s)\psi_2(J_4(s))ds$, hence $Y_4(0) = J_4(0) = u_0^q$, and $J_4^q(t) \leq Y_4(t)$ but $q \geq 1$, thus

$$J_4(t) \leq Y_4(t) \qquad \forall t \in I. \tag{2.43}$$

Differentiating $Y_4(t)$ with respect t and using (2.42), (2.43), leads to

$$\frac{dY_4(t)}{dt} \leq q\alpha'(t) f(\alpha(t)) \psi_1(Y_4(t)) Y_4^{[p+q-1]}(t) + \alpha'(t) g(\alpha(t)) \psi_2(Y_4(t)), \tag{2.44}$$

for all $t \in I$, since $\psi_2(Y_4(t)) > 0, \forall t > 0$, from (2.44) we get

$$\frac{dY_4(t)}{\psi_2(Y_4(t))} \leq q\alpha'(t) f(\alpha(t)) \frac{\psi_1(Y_4(t)) Y_4^{[p+q-1]}(t)}{\psi_2(Y_4(t))} dt + \alpha'(t) g(\alpha(t)) dt, \forall t \in I.$$

By taking $t = s$ in the last inequality and integrating both sides from 0 to t, and using (2.39), we have

$$\Psi_2(Y_4(t)) \leq \Psi_2(u_0^q) + \int_0^t q\alpha'(s) f(\alpha(s)) \frac{\psi_1(Y_4(s)) Y_4^{[p+q-1]}(s)}{\psi_2(Y_4(s))} ds + \int_0^t \alpha'(s) g(\alpha(s)) ds, \qquad \forall t \in I,$$

where Ψ_1 is defined by (2.39) from the above inequality we have

$$\Psi_2(Y_4(t)) \leq \Psi_2(u_0^q) + \int_0^T \alpha'(s) g(\alpha(s)) ds + \int_0^t q\alpha'(s) f(\alpha(s)) \frac{\psi_1(Y_4(s)) Y_4^{[p+q-1]}(s)}{\psi_2(Y_4(s))} ds,$$

for all $t < T$, where $T \in [0, T_1]$ is chosen arbitrary. Let $Y_5(t)$ denote the function on the right-hand side of the above inequality, which is a positive and nondecreasing function on I with $Y_5(t) = \Psi_2(u_0^q) + \int_0^T \alpha'(s) g(\alpha(s)) ds$ and

$$Y_4(t) \leq \Psi_1^{-1}(Y_5(t)), \forall t < T. \tag{2.45}$$

Differentiating $Y_5(t)$ with respect to t, and using (2.45) we obtain

$$\begin{aligned} \frac{dY_5(t)}{dt} &\leq q\alpha'(t) f(\alpha(t)) \frac{\psi_1(Y_3(t)) Y_4^{[p+q-1]}(t)}{\psi_2(Y_4(t))} \\ &\leq q\alpha'(t) f(\alpha(t)) \frac{\psi_1(\psi_2(Y_5(t)))(\psi_2(Y_5(t)))^{[p+q-1]}(t)}{\psi_2(\psi_2(Y_5(t)))}, \forall t < T. \end{aligned}$$

From the above inequality, by the definition of Ψ_2 in (2.40) and let $t = T$ we have

$$\Psi_2(Y_5(t)) \leq \Psi_2\left(\Psi_1(u_0^q) + \int_0^{\alpha(T)} g(s) ds\right) + \int_0^{\alpha(T)} q f(s) ds, \forall t < T. \tag{2.46}$$

Since $0 < T < T_1$ is chosen, from (2.41) , (2.43), (2.45) and (2.46), we get the required inequality in (2.38). The proof is completed. □

Remark 2.7. If $q = 1$, then Theorem 2.5 reduces to Theorem 4 in [16].

Remark 2.8. If $\psi_1(t) = \psi_2(t) = t$, then Theorem 2.5 reduces to Theorem 2.3.

3 Application

In this section we present some applications of the inequalities given in our Theorem 2.1 and Theorem 2.5 in order to illustrate the usefulness of our results.

Now we give an example for the application of our Theorem 2.1 to the following retarded integro-differential equation

Example 3.1 Consider the retarded integro-differential equation :

$$\begin{cases} pu^{p-1}(t)\frac{du(t)}{dt} = M(t, u(\alpha(t)), \int_0^t H(s, u(\alpha(s)))ds), \forall t \in I, \\ u(0) = u_0, \end{cases} \tag{3.1}$$

where $M \in \mathcal{C}(I \times \mathbb{R}^3, \mathbb{R})$, $H \in \mathcal{C}(I \times \mathbb{R}, \mathbb{R})$, $|u_0| > 0$ is a constant, satisfy the following conditions

$$|H(t, u(t))| \leq g(t)|u(t)|, \tag{3.2}$$

$$|M(t, u(\alpha(t)), \int_0^t H(s, u(\alpha(s)))ds)| \leq f(t)|\Big(|u(t)|^p + \int_0^t |K(s, u(\alpha(s)))|ds\Big), \tag{3.3}$$

$$u_0^{\frac{1}{p}}\left[1 + \int_0^{\alpha(t)} \frac{f(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))} \exp\left(\int_0^s \left[\frac{[f(\alpha^{-1}(\lambda)) + g(\alpha^{-1}(\lambda))]}{\alpha'(\alpha^{-1}(\lambda))}\right] d\lambda\right) ds\right]^{\frac{1}{p}} < \infty, \tag{3.4}$$

where $u(t), f(t), g(t), \alpha(t), p$ as defined in Theorem 2.1. Integrate both sides of the equation (3.1) from 0 to t, we have

$$u^p(t) = u_0 + \int_0^t M(s, u(\alpha(s)), \int_0^s H(\lambda, u(\alpha(\lambda)))d\lambda) ds, \forall t \in I, \tag{3.5}$$

using the conditions (3.2) and (3.3), from (3.5) we get

$$\begin{aligned} |u(t)|^p &\leq |u_0| + \int_0^t f(s)\left[|u(s)|^p + \int_0^s g(\lambda)|u(\lambda)|d\lambda\right] ds, \\ &\leq |u_0| + \int_0^{\alpha(t)} \frac{f(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))}\left[|u(s)|^p + \int_0^s \frac{g(\alpha^{-1}(\lambda))}{\alpha'(\alpha^{-1}(\lambda))}|u(\lambda)|d\lambda\right] ds \end{aligned}$$

Now, a suitable application of the inequality given in Theorem 2.1 to the above inequality yields

$$|u(t)| \leq u_0^{\frac{1}{p}}\left[1 + \int_0^{\alpha(t)} \frac{f(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))} \exp\left(\int_0^s \left[\frac{[f(\alpha^{-1}(\lambda)) + g(\alpha^{-1}(\lambda))]}{\alpha'(\alpha^{-1}(\lambda))}\right] d\lambda\right) ds\right]^{\frac{1}{p}}, \tag{3.6}$$

for all $t \in I$. Thus, from the hypotheses (3.4) and the estimation in (3.6), implies the boundedness of the solution $u(t)$ of (3.1).

Now we give an example for the application of our Theorem 2.5 to the following retarded integro-differential equation

$$\frac{du(t)}{dt} = H(t, u(\alpha(t)), \int_0^t K(s, u(\alpha(s)))ds), \forall t \in I, \tag{3.7}$$

with initial condition $u(0) = u_0$, we assume that $K \in \mathcal{C}(\mathbb{R} \times \mathbb{R}, \mathbb{R})$, $H \in \mathcal{C}(\mathbb{R}^*, \mathbb{R})$, $|u_0| > 0$ is a constant, satisfy the following conditions

$$|K(t, u(t))| \leq g(t)\psi_2(|u(t)|), \tag{3.8}$$

$$|H(t, u(\alpha), \int_0^t K(s, u(\alpha))ds)| \leq f(t)\psi_1(|u(\alpha)|)\Big(|u(t)|^q + \int_0^t |K(s, u(\alpha))|ds\Big)^p, \tag{3.9}$$

where $f(t), g(t) \in \mathcal{C}(I, I)$.

Example 3.2 Consider the nonlinear retarded integro-differential equation (3.7) and suppose that K, H satisfy the conditions (3.8) and (3.9), and $\psi_1(t), \psi_2(t)$ are nondecreasing and continuous functions defined on I with $\psi_i(t) > 0, \forall t > 0, i = 1, 2$, and $\alpha(t) \in \mathcal{C}^1(I, I)$ is a nondecreasing function with $\alpha(t) \leq t, \alpha(0) = 0$ then all the solutions of the equation (3.7) exist on I and satisfy the following estimation

$$|u(t)| \leq \Psi_1^{-1}\Bigg[\Psi_2^{-1}\Bigg(\Psi_2\Bigg(\Psi_1(|u_0^q|) + \int_0^{\alpha(t)} \frac{g(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))}ds\Bigg) + \int_0^{\alpha(t)} \frac{f(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))}ds\Bigg)\Bigg], \tag{3.10}$$

for all $t < T_2$, where

$$\Psi_1(r) = \int_1^r \frac{dt}{\psi_2(t)}, r > 0, \tag{3.11}$$

$$\Psi_2(r) = \int_1^r \frac{\Psi_1(\Psi_1^{-1}(s))ds}{\psi_1(\Psi_1^{-1}(s))(\Psi_1^{-1}(s))^{p+q-1}}, r > 0, \tag{3.12}$$

and T_2 is the largest number such that

$$\begin{aligned}\Psi_2\Bigg(\Psi_1(|u_0|^q) + \int_0^{\alpha(t)} \frac{g(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))}ds\Bigg)\Bigg) + \int_0^{\alpha(t)} \frac{f(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))}ds \\ \leq \frac{\Psi_1(\Psi_1^{-1}(s))ds}{\psi_1(\Psi_1^{-1}(s))(\Psi_1^{-1}(s))^{p+q-1}},\end{aligned}$$

$$\Psi_2^{-1}\Bigg(\Psi_2\Bigg(\Psi_1(|u_0|^q) + \int_0^{\alpha(t)} \frac{g(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))}ds\Bigg) + \int_0^{\alpha(t)} \frac{f(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))}ds\Bigg) \leq \frac{dt}{\psi_2(t)}.$$

Proof. Integrating both side of the Eq. (3.7) from 0 to t, we have

$$u(t) = u_0 + \int_0^t H(s, u(\alpha(s)), \int_0^s K(\lambda, u(\alpha(\lambda)))ds), \forall t \in I, \tag{3.13}$$

using the conditions (3.8) and (3.9), from (3.13) we get

$$|u(t)| \leq |u_0| + \int_0^t f(s)\varphi(|u(\alpha(s))|)\Bigg[|u^q(s)| + \int_0^s g(\lambda)\varphi(|u(\lambda)|)d\lambda\Bigg]^p ds, \tag{3.14}$$

the inequality (3.14) can be written in the form

$$\begin{aligned}|u(t)| \leq &|u_0| + \int_0^{\alpha(t)} \frac{f(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))}\varphi(|u(\alpha(s))|) \\ &\times \Bigg[|u^q(s)| + \int_0^s \frac{g(\alpha^{-1}(\lambda))}{\alpha'(\alpha^{-1}(\lambda))}\varphi(|u(\lambda)|)d\lambda\Bigg]^p ds, \forall t \in I.\end{aligned} \tag{3.15}$$

Now, a suitable application of the inequality given in Theorem 2.5 to (3.15), we get the required inequality in (3.10). The proof is completed. □

4 Conclusions

a : In the section 1, a brief historical evolution of integral inequalities and some results of El-Owaidy et al. [1] has been presented.

b : In the section 2, we have studied several new retarded integral inequalities by generalize some results which established in El-Owaidy et al. [1].

c : In the section 3, we have presented some applications for some inequalities given in section 2 in order to illustrate the usefulness of our results.

d : Authors think that, the rest of the integral inequalities which established in El-Owaidy et al. [1], can be developed to the retarded integral inequalities by the same techniques.

e : We finally mention that the retarded integral inequalities obtained in this paper allow us to study the stability, boundedness and asympototic behaviour of the solutions of a class of more general retarded nonlinear differential, integral and integro-differential equations.

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Competing Interests

The authors declare that no competing interests exist.

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