British Journal of Mathematics & Computer Science
5(4): 492-501, 2015, Article no.BJMCS.2015.035
ISSN: 2231-0851

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Existence of Nonoscillatory Solutions of Higher Order Forced Neutral Dynamic Equations with Time Delay on Time Scales

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Article Information
DOI: 10.9734/BJMCS/2015/14006
Editor(s):
(1) Sheng Zhang, Department of Mathematics, Bohai University, Jinzhou, China.
Reviewers:
(1) Anonymous, Jiangnan University, China.
(2) Anonymous, Qingdao Technological University, China.
Complete Peer review History:
http://www.sciencedomain.org/review-history.php?iid=728&id=6&aid=6917

Original Research Article

Received: 14 September 2014
Accepted: 15 October 2014
Published: 14 November 2014

Abstract

The existence of nonoscillatory solutions of a class of higher order forced neutral dynamic equations with time delay on time scales is discussed. The main tool is the Banach fixed point theorem. Based on the different values of $p(t)$, we give several existence theorems of nonoscillatory solutions of our discussing equations. An example is also presented to illustrate the applications of the obtained results.

Keywords: Time scales; time delay; neutral equations; dynamic equations; nonoscillatory solution.
2010 Mathematics Subject Classification: 34C10; 34C15

1 Introduction

Theory of dynamic equations on time scales has a very important theoretical significance and broad application prospects [1,2], and the study of dynamic equations on time scales can further improve the related results of differential and difference equations with time delay. Therefore, the study of

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Project supported by NSFN of China (No.11171192) and NSF of Shandong Province (No. ZR2013AM005).

dynamic equations on time scales has become one of the hot issues in recent years and the calculus theory on time scales is more and more perfect [3,4]. Certain results have been obtained in the qualitative theory and the existence of nonoscillatory solutions of dynamical equations on time scales [5,6,7,8]. There have been some research results [9,10,11] about the oscillation and nonoscillation of higher order neutral differential equation, but the achievements about the existence of nonoscillatory solutions of the higher order neutral differential equations with time delay are less.

In [12] authors studied the neutral dynamic equation with positive and negative coefficients on time scales

$$(x(t) - x(t-\tau))^{\Delta} + p(t)x(t-\theta) - Q(t)x(t-\delta) = 0,$$

and obtained the sufficient conditions for the existence of the bounded positive solution and bounded oscillatory solution of this equation.

In [13] authors discussed the following dynamic equation by $Krasnoselskii$ fixed point theorem

$$(x(t) + p(t)x(g(t)))^{\Delta} + f(t, x(h(t))) = 0,$$

and obtained the criteria for the existence of nonoscillatory solution of this equation.

In [14] authors considered the n-order neutral differential equation

$$[x(t) + px(t-\tau)]^{(n)} + f[t, x(t-\tau_1(t)), \cdots, x(t-\tau_k(t))] = 0,$$

and gained the sufficient conditions for the existence of the positive solution of this equation as $p \neq 1$ and the necessary and sufficient conditions for the existence of the bounded positive solution of this equation as $p = -1$.

In this paper, we consider the existence of nonoscillatory solution of the following forced n-order neutral dynamic equation with time delay on time scales

$$[x(t) + p(t)x(\tau(t))]^{\Delta^n} + f[t, x(\delta_1(t)), x(\delta_2(t)), \cdots, x(\delta_m(t))] = g(t), \; t \in [t_0, \infty)_{\mathbb{T}}, \tag{1.1}$$

where $\mathbb{T}$ is an no upper-bounded time scale, $p, g \in C_{rd}([t_0, \infty)_{\mathbb{T}}, R), \tau, \delta_i \in C_{rd}([t_0, \infty)_{\mathbb{T}}, [t_0, \infty)_{\mathbb{T}}), i = 1, 2, \cdots, m, f \in C([t_0, \infty)_{\mathbb{T}} \times R^m, R), t_0 \in \mathbb{T}$, and C_{rd} represents the set of all right dense continuous functions.

According to the different values of $p(t)$, we construct different special nonempty closed sets and special operators in Banach spaces, and prove the existence of nonoscillatory solution of (1.1) by using Banach fixed point theorem and extend the previous results. By giving special time scales, we obtain the existence of nonoscillatory solution of the forced n-order neutral differential equation with time delay. An example is also presented to illustrate the applications of the obtained results.

2 Preliminaries and Lemmas

A time scale is an arbitrary nonempty closed subset of the real numbers. Let $\mathbb{T}$ be a time scale. For $t \in \mathbb{T}$ we define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$, while the back jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$, where $\inf \emptyset = \sup \mathbb{T}$ and $\sup \emptyset = \inf \mathbb{T}$. If $\sigma(t) > t$, we say that $t \in \mathbb{T}$ is right-scattered, while if $\rho(t) < t$ we say that t is left-scattered. Also, if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then t is called right-dense, and if $t > \inf \mathbb{T}$ and $\rho(t) = t$, then t is called left-dense. Points that are right-dense and left-dense at the same time are called dense. Defining the set $\mathbb{T}^{\kappa}$ as following: If $\mathbb{T}$ has a maximum left-scattered point M, then $\mathbb{T}^{\kappa} = \mathbb{T} - \{M\}$. Otherwise, $\mathbb{T}^{\kappa} = \mathbb{T}$. Assume $f : \mathbb{T} \to R$, $t \in \mathbb{T}^{\kappa}$. If there is a constant α, for any $\varepsilon > 0$, and there is a neighborhood $U_{\mathbb{T}} (= (t-\delta, t+\delta) \bigcap \mathbb{T}$ for some $\delta > 0)$ of t, such that

$$|f(\sigma(t)) - f(s) - \alpha[\sigma(t) - s]| \leq \varepsilon |\sigma(t) - s|, s \in U_{\mathbb{T}},$$

then we say that f is Δ differentiable at t, and the derivative is α, which is denoted by $f^{\Delta}(t)$. Similarly, we can define n order Δ derivative $f^{\Delta^n} = (f^{\Delta^{n-1}})^{\Delta}$. Other concepts and calculations on time scales can be seen in [1-2].

Lemma 2.1 [10] (Banach fixed point theorem) Suppose (X, ρ) is a complete metric space, Ω is a nonempty closed subset of X and T is a contraction mapping from Ω to itself. Then T has a unique fixed point in Ω.

Lemma 2.2 [15] Let $a \in \mathbb{T}^{\kappa}, b \in \mathbb{T}, a < b,\ t \in \mathbb{T}^{\kappa}, t > a$. Assume $f : \mathbb{T} \times \mathbb{T}^{\kappa} \to R$ is continuous and $f^{\Delta}(t, \cdot)$ is rd-continuous in $[a, \sigma(t)]$(f^{Δ} denotes the derivative of the first variable). Set $h(t) := \int_t^b f(t, \tau)\Delta\tau$, then $h^{\Delta}(t) = \int_t^b f^{\Delta}(t, \tau)\Delta\tau - f(\sigma(t), t)$.

We say that a nontrivial solution of (1.1) is oscillatory if it is finally not positive or not negative, otherwise we say that it is nonoscillatory. If all solutions of (1.1) are oscillatory, then we say that the equation (1.1) is oscillatory.

For convenience, we list the following assumptions:

(H_1) $\delta_i(t) \le t, \forall t \in [t_0, \infty)_{\mathbb{T}}$, and $\lim\limits_{t \in \mathbb{T}, t \to \infty} \delta_i(t) = \infty, i = 1, 2, \cdot\cdot, m$;

(H_2) $\tau(t) \le t, \forall t \in [t_0, \infty)_{\mathbb{T}}$, and $\lim\limits_{t \in \mathbb{T}, t \to \infty} \tau(t) = \infty$;

(H_3) $\exists G \in C^n_{rd}([t_0, \infty)_{\mathbb{T}}, R), G_0 \in R, s.t.\ G^{\Delta^n}(t) = g(t), \forall t \in [t_0, \infty)_{\mathbb{T}}, \lim\limits_{t \in \mathbb{T}, t \to \infty} G(t) = G_0$;

(H_4) $x_i f(t, x_1, x_2, \cdot\cdot, x_m) \ge 0, f(t, x_1, x_2, \cdot\cdot, x_m)$ is not decreasing for each $x_i \in R, i = 1, 2, \cdot\cdot\cdot, m, t \in [t_0, \infty)_{\mathbb{T}}$, and $\exists b > 2, s.t. \int_{t_0}^{\infty} (\sigma(s) - t_0)^{n-1} f(s, b, \cdot\cdot, b)\Delta s < \infty$.

3 Main Results

We consider the Banach space

$$BC[t_0, \infty)_{\mathbb{T}} = \{x : x \in C_{rd}([t_0, \infty)_{\mathbb{T}}, R), \sup_{t \in [t_0, \infty)_{\mathbb{T}}} |x(t)| < \infty\}$$

endowed with the norm

$$\| x \| = \sup_{t \in [t_0, \infty)_{\mathbb{T}}} |x(t)|.$$

The main results in this paper are the following theorems.

Theorem 3.1 Assume $(H_1) \sim (H_4)$ hold, and there exist $0 < c_1 < c_2 < 1$, such that $c_1 \le p(t) \le c_2$, and for all $0 \le u_i, v_i \le b, i = 1, 2, \cdot\cdot\cdot, m$, $| f(t, v_1, v_2, \cdot\cdot\cdot, v_m) - f(t, u_1, u_2, \cdot\cdot\cdot, u_m) | \le m f(t, b, b, \cdot\cdot, b) \sup\limits_{1 \le i \le m} | v_i - u_i |$ holds. Then there exists a bounded nonoscillatory solution of (1.1).

Proof Let $\Omega = \{x : x \in BC[t_0, \infty)_{\mathbb{T}}, \frac{b(1 - c_2)}{2(1 - c_1)} \le x(t) \le b, \forall t \in [t_0, \infty)_{\mathbb{T}}\}$. Clearly, Ω is a nonempty closed subset in $BC[t_0, \infty)_{\mathbb{T}}$. From $(H_1) \sim (H_4)$, there exists sufficient large $T_0 \in [t_0, \infty)_{\mathbb{T}}$, when $t \in [T_0, \infty)_{\mathbb{T}}$,

$$\tau(t) \ge t_0, \delta_i(t) \ge t_0, i = 1, 2, \cdot\cdot\cdot, m, \tag{3.1}$$

$$| G(t) - G_0 | \le \frac{(b - 2)(1 - c_2)}{4}, \tag{3.2}$$

$$\frac{1}{(n-1)!} \int_{T_0}^{\infty} (\sigma(s) - t_0)^{n-1} f(s, b, \cdot\cdot\cdot, b)\Delta s \le \frac{1 - c_2}{2m}. \tag{3.3}$$

Define the operator $S : \Omega \to BC[t_0, \infty)_{\mathbb{T}}$ as follows

$$(Sx)(t) = \begin{cases} \frac{b}{4(1 - c_1)}(3 + c_2 - c_1 - 3c_1c_2) - p(t)x(\tau(t)) + \\ \frac{(-1)^{n-1}}{(n-1)!} \int_t^{\infty} (\sigma(s) - t)^{n-1} f(s, x(\delta_1(s)), x(\delta_2(s)), \cdot\cdot\cdot, x(\delta_m(s)))\Delta s + G(t) - G_0, t \in [T_0, \infty)_{\mathbb{T}}; \\ (Sx)(T_0), t \in [t_0, T_0)_{\mathbb{T}}. \end{cases}$$

Firstly, we prove $S\Omega \subseteq \Omega$. When $t \in [t_0, T_0)_{\mathbb{T}}$, $(Sx)(t) = (Sx)(T_0)$, so we only need to discuss the case on $t \in [T_0, \infty)_{\mathbb{T}}$.

From (H_4) and $(3.1) \sim (3.3)$, for all $t \in [T_0, \infty)_{\mathbb{T}}, x \in \Omega$, we have

$$\begin{aligned}(Sx)(t) \geq\ & \frac{b}{4(1-c_1)}(3 + c_2 - c_1 - 3c_1c_2) - c_2 b - \frac{1}{(n-1)!}\int_{T_0}^{\infty}(\sigma(s) - t_0)^{n-1} f(s, b, \cdots, b)\Delta s \\ & - \mid G(t) - G_0 \mid \\ \geq\ & \frac{b}{4(1-c_1)}(3 + c_2 - c_1 - 3c_1c_2) - c_2 b - \frac{1}{2}(1 - c_2) - \frac{1}{4}(b-2)(1-c_2) = \frac{b(1-c_2)}{2(1-c_1)}, \\ (Sx)(t) \leq\ & \frac{b}{4(1-c_1)}(3 + c_2 - c_1 - 3c_1c_2) - \frac{bc_1(1-c_2)}{2(1-c_1)} + \frac{1}{(n-1)!}\int_{T_0}^{\infty}(\sigma(s) - t_0)^{n-1} f(s, b, \cdots, b)\Delta s \\ & + \mid G(t) - G_0 \mid \\ \leq\ & \frac{b}{4(1-c_1)}(3 + c_2 - c_1 - 3c_1c_2) - \frac{bc_1(1-c_2)}{2(1-c_1)} + \frac{1}{2}(1-c_2) + \frac{1}{4}(b-2)(1-c_2) = b.\end{aligned}$$

Therefore $\frac{b(1-c_2)}{2(1-c_1)} \leq Sx(t) \leq b$, so $S\Omega \subseteq \Omega$.

Secondly, we prove that S is a contraction mapping on Ω. When $t \in [t_0, T_0)_{\mathbb{T}}$, $\mid (Sx)(t) - (Sy)(t) \mid = \mid (Sx)(T_0) - (Sy)(T_0) \mid$, so we only need to discuss the case on $t \in [T_0, \infty)_{\mathbb{T}}$.

For all $t \in [T_0, \infty)_{\mathbb{T}}, x, y \in \Omega$, from (H_4) and $(3.1), (3.3)$,

$$\begin{aligned}\mid (Sx)(t) - (Sy)(t) \mid \leq\ & p(t) \mid x(\tau(t)) - y(\tau(t)) \mid + \frac{1}{(n-1)!}\int_t^{\infty}(\sigma(s) - t)^{n-1} \mid f(s, x(\delta_1(s)) \\ & , \cdots, x(\delta_m(s))) - f(s, y(\delta_1(s)), \cdots, y(\delta_m(s))) \mid \Delta s \\ \leq\ & c_2 \parallel x - y \parallel + \frac{1-c_2}{2} \parallel x - y \parallel = \frac{1+c_2}{2} \parallel x - y \parallel,\end{aligned}$$

so $\parallel Sx - Sy \parallel \leq \frac{1+c_2}{2} \parallel x - y \parallel$. Thus S is a contraction mapping on Ω since $0 < c_2 < 1$.

By Lemma 2.1, there exists $x \in \Omega$, such that $(Sx)(t) = x(t)$ holds for all $t \in [t_0, \infty)_{\mathbb{T}}$. Therefore, we have

$$\begin{aligned}x(t) + p(t)x(\tau(t)) =\ & \frac{b}{4(1-c_1)}(3 + c_2 - c_1 - 3c_1c_2) \\ & + \frac{(-1)^{n-1}}{(n-1)!}\int_t^{\infty}(\sigma(s) - t)^{n-1} f(s, x(\delta_1(s)), x(\delta_2(s)), \cdots, x(\delta_m(s)))\Delta s + G(t) - G_0.\end{aligned}$$

By Lemma 2.2, we obtain

$$[x(t) + p(t)x(\tau(t))]^{\Delta} = \frac{(-1)^n}{(n-2)!}\int_t^{\infty}(\sigma(s) - t)^{n-2} f(s, x(\delta_1(s)), x(\delta_2(s)), \cdots, x(\delta_m(s)))\Delta s + G^{\Delta}(t),$$

$$[x(t) + p(t)x(\tau(t))]^{\Delta^2} = \frac{(-1)^{n+1}}{(n-3)!}\int_t^{\infty}(\sigma(s) - t)^{n-3} f(s, x(\delta_1(s)), x(\delta_2(s)), \cdots, x(\delta_m(s)))\Delta s + G^{\Delta^2}(t),$$

. . . .

$$[x(t) + p(t)x(\tau(t))]^{\Delta^{n-1}} = (-1)^{2n-2}\int_t^{\infty} f(s, x(\delta_1(s)), x(\delta_2(s)), \cdots, x(\delta_m(s)))\Delta s + G^{\Delta^{n-1}}(t),$$

$$[x(t) + p(t)x(\tau(t))]^{\Delta^n} + f[t, x(\delta_1(t)), x(\delta_2(t)), \cdots, x(\delta_m(t))] = g(t).$$

Therefore $x(t)$ is a bounded nonoscillatory solution of (1.1). □

Theorem 3.2 Assume $(H_1) \sim (H_4)$ hold, and there exist $-1 < p_1 < p_2 < 0$, such that $p_1 \leq p(t) \leq p_2$, and for all $0 \leq u_i, v_i \leq b, i = 1, 2, \cdots, m, | f(t, v_1, v_2, \cdots, v_m) - f(t, u_1, u_2, \cdots, u_m) | \leq mf(t, b, b, \cdots, b) \sup_{1\leq i\leq m} | v_i - u_i |$ holds. Then there exists a bounded nonoscillatory solution of (1.1).

Proof Let $\Omega = \{x : x \in BC[t_0, \infty)_{\mathbb{T}}, \frac{b(1 + p_1)}{2(1 + p_2)} \leq x(t) \leq b, \forall t \in [t_0, \infty)_{\mathbb{T}}\}$. Obviously, Ω is a nonempty closed subset in $BC[t_0, \infty)_{\mathbb{T}}$. From $(H_1) \sim (H_4)$, there exists sufficient large $T_0 \in [t_0, \infty)_{\mathbb{T}}$, when $t \in [T_0, \infty)_{\mathbb{T}}$,

$$\tau(t) \geq t_0, \delta_i(t) \geq t_0, i = 1, 2, \cdots, m, \tag{3.4}$$

$$| G(t) - G_0 | \leq \frac{(b - 2)(1 + p_1)}{4}, \tag{3.5}$$

$$\frac{1}{(n - 1)!} \int_{T_0}^{\infty} (\sigma(s) - t_0)^{n-1} f(s, b, \cdots, b)\Delta s \leq \frac{1 + p_1}{2m}. \tag{3.6}$$

Define the operator $S : \Omega \to BC[t_0, \infty)_{\mathbb{T}}$ as follows

$$(Sx)(t) = \begin{cases} \frac{3b(1 + p_1)}{4} - p(t)x(\tau(t))+ \\ \frac{(-1)^{n-1}}{(n - 1)!} \int_t^{\infty} (\sigma(s) - t)^{n-1} f(s, x(\delta_1(s)), x(\delta_2(s)), \cdots, x(\delta_m(s)))\Delta s + G(t) - G_0, t \in [T_0, \infty)_{\mathbb{T}}; \\ (Sx)(T_0), t \in [t_0, T_0)_{\mathbb{T}}. \end{cases}$$

Firstly, we prove $S\Omega \subseteq \Omega$. From (H_4) and $(3.4) \sim (3.6)$, for all $t \in [T_0, \infty)_{\mathbb{T}}, x \in \Omega$, we have

$$\begin{aligned} (Sx)(t) &\geq \frac{3b(1 + p_1)}{4} - p_2\frac{b(1 + p_1)}{2(1 + p_2)} - \frac{1}{(n - 1)!} \int_{T_0}^{\infty} (\sigma(s) - t_0)^{n-1} f(s, b, \cdots, b)\Delta s \\ &\quad - | G(t) - G_0 | \\ &\geq \frac{3b(1 + p_1)}{4} - p_2\frac{b(1 + p_1)}{2(1 + p_2)} - \frac{1}{2}(1 + p_1) - \frac{(b - 2)(1 + p_1)}{4} = \frac{b(1 + p_1)}{2(1 + p_2)}, \\ (Sx)(t) &\leq \frac{3b(1 + p_1)}{4} - p_1 b + \frac{1}{(n - 1)!} \int_{T_0}^{\infty} (\sigma(s) - t_0)^{n-1} f(s, b, \cdots, b)\Delta s \\ &\quad + | G(t) - G_0 | \\ &\leq \frac{3b(1 + p_1)}{4} - p_1 b + \frac{1}{2}(1 + p_1) + \frac{(b - 2)(1 + p_1)}{4} = b. \end{aligned}$$

Therefore

$$\frac{b(1 + p_1)}{2(1 + p_2)} \leq (Sx)(t) \leq b,$$

so $S\Omega \subseteq \Omega$.

Secondly, we prove S is a contraction mapping on Ω. For all $t \in [T_0, \infty)_{\mathbb{T}}, x, y \in \Omega$, from (H_4) and $(3.4), (3.6)$, we obtain

$$\begin{aligned} | (Sx)(t) - (Sy)(t) | &\leq | p(t) || x(\tau(t)) - y(\tau(t)) | + \frac{1}{(n - 1)!} \int_t^{\infty} (\sigma(s) - t)^{n-1} | f(s, x(\delta_1(s)) \\ &\quad , \cdots, x(\delta_m(s))) - f(s, y(\delta_1(s)), \cdots, y(\delta_m(s))) | \Delta s \\ &\leq -p_1 \| x - y \| + \frac{1 + p_1}{2} \| x - y \| = \frac{1 - p_1}{2} \| x - y \|, \end{aligned}$$

so $\| Sx - Sy \| \leq \frac{1 - p_1}{2} \| x - y \|$. Hence S is a contraction mapping on Ω since $-1 < p_1 < 0$.

By Lemma 2.1, there exists $x \in \Omega$ and $(Sx)(t) = x(t)$ holds for all $t \in [t_0, \infty)_{\mathbb{T}}$.
Finally, by Lemma 2.2 and simple calculation, $x(t)$ is a bounded nonoscillatory solution of (1.1).□

Theorem 3.3 Suppose that $(H_1) \sim (H_4)$ hold, and τ is a monotone increasing mapping, and there exist $d_1 < d_2 < -1, s.t.\ (1-b)d_2 + d_1 > 0, d_1 \le p(t) \le d_2$, and for all $0 \le u_i, v_i \le b, i = 1, 2, \cdots, m, | f(t, v_1, v_2, \cdots, v_m) - f(t, u_1, u_2, \cdots, u_m) | \le mf(t, b, b, \cdots, b) \sup\limits_{1\le i\le m} | v_i - u_i |$ holds. Then there exists a bounded nonoscillatory solution of (1.1).

Proof Let $\Omega = \{x : x \in BC[t_0, \infty)_{\mathbb{T}}, \frac{b(1+d_2)}{2(1+d_1)} \le x(t) \le b, \forall t \in [t_0, \infty)_{\mathbb{T}}\}$. Then Ω is a nonempty closed subset in $BC[t_0, \infty)_{\mathbb{T}}$. From $(H_1) \sim (H_4)$, there exists sufficient large $T_0 \in [t_0, \infty)_{\mathbb{T}}$, when $t \in [T_0, \infty)_{\mathbb{T}}$,

$$\tau(t) \ge t_0, \tau^{-1}(t) \ge t_0, \delta_i(t) \ge t_0, i = 1, 2, \cdots, m, \tag{3.7}$$

$$| G(t) - G_0 | \le \frac{(d_2+1)(d_1+d_2-bd_2)}{2(d_1+d_2)}, \tag{3.8}$$

$$\frac{1}{(n-1)!}\int_{\tau^{-1}(T_0)}^{\infty} (\sigma(s) - t_0)^{n-1} f(s, b, \cdots, b)\Delta s \le \frac{-1-d_2}{2m}. \tag{3.9}$$

Define the operator $S : \Omega \to BC[t_0, \infty)_{\mathbb{T}}$ as follows

$$(Sx)(t) = \begin{cases} \frac{1}{p(\tau^{-1}(t))}[\frac{b(d_2+1)}{d_1+d_2}(d_1 + \frac{d_2}{2}) - x(\tau^{-1}(t)) + \\ \frac{(-1)^{n-1}}{(n-1)!}\int_{\tau^{-1}(t)}^{\infty} (\sigma(s) - \tau^{-1}(t))^{n-1} f(s, x(\delta_1(s)), \cdots, x(\delta_m(s)))\Delta s + G(\tau^{-1}(t)) - G_0], t \in [T_0, \infty)_{\mathbb{T}}; \\ (Sx)(T_0), t \in [t_0, T_0)_{\mathbb{T}}. \end{cases}$$

Firstly, we prove $S\Omega \subseteq \Omega$. From (H_4) and $(3.7) \sim (3.9)$, for all $t \in [T_0, \infty)_{\mathbb{T}}, x \in \Omega$,

$$\begin{aligned}(Sx)(t) &\ge \frac{1}{d_1}[\frac{b(d_2+1)}{d_1+d_2}(d_1 + \frac{d_2}{2})] - \frac{b(1+d_2)}{2d_1(1+d_1)} \\ &+ \frac{1}{p(\tau^{-1}(t))}\frac{1}{(n-1)!}\int_{\tau^{-1}(T_0)}^{\infty} (\sigma(s) - t_0)^{n-1} f(s, b, \cdots, b)\Delta s + \frac{1}{p(\tau^{-1}(t))} | G(t) - G_0 | \\ &\ge \frac{1}{d_1}[\frac{b(d_2+1)}{d_1+d_2}(d_1 + \frac{d_2}{2})] - \frac{1}{d_1}\frac{b(1+d_2)}{2(1+d_1)} - \frac{d_2+1}{2d_2} + \frac{1}{d_2}\frac{(d_2+1)(d_1+d_2-bd_2)}{2(d_1+d_2)} = \frac{b(1+d_2)}{2(1+d_1)},\end{aligned}$$

$$\begin{aligned}(Sx)(t) &\le \frac{1}{d_2}[\frac{b(d_2+1)}{d_1+d_2}(d_1 + \frac{d_2}{2})] - \frac{1}{d_2}b - \frac{1}{p(\tau^{-1}(t))}\frac{1}{(n-1)!}\int_{\tau^{-1}(T_0)}^{\infty} (\sigma(s) - t_0)^{n-1} f(s, b, \cdots, b)\Delta s \\ &- \frac{1}{p(\tau^{-1}(t))} | G(t) - G_0 | \\ &\le \frac{1}{d_2}[\frac{b(d_2+1)}{d_1+d_2}(d_1 + \frac{d_2}{2})] - \frac{1}{d_2}b + \frac{d_2+1}{2d_2} - \frac{1}{d_2}\frac{(d_2+1)(d_1+d_2-bd_2)}{2(d_1+d_2)} = b.\end{aligned}$$

So

$$\frac{b(1+d_2)}{2(1+d_1)} \le x(t) \le b.$$

Therefore $S\Omega \subseteq \Omega$.

Secondly, we prove S is a contraction mapping on Ω. For all $t \in [T_0, \infty)_{\mathbb{T}}, x, y \in \Omega$, from (H_4) and $(3.7), (3.9)$, we obtain

$$| (Sx)(t) - (Sy)(t) | \le | \frac{1}{p(\tau^{-1}(t))} | [| x(\tau^{-1}(t)) - y(\tau^{-1}(t)) | + \frac{1}{(n-1)!}\int_{\tau^{-1}(T_0)}^{\infty} (\sigma(s) - \tau^{-1}(t))^{n-1}$$

$$| f(s, x(\delta_1(s)), \cdots, x(\delta_m(s))) - f(s, y(\delta_1(s)), \cdots, y(\delta_m(s))) | \Delta s \,]$$

$$\leq \frac{1}{-d_2} \parallel x - y \parallel + \frac{-1-d_2}{-2d_2} \parallel x - y \parallel = \frac{d_2 - 1}{2d_2} \parallel x - y \parallel.$$

Hence $\parallel Sx - Sy \parallel \leq \frac{d_2-1}{2d_2} \parallel x - y \parallel$. Thus S is a contraction mapping on Ω since $d_2 < -1$,.

From Lemma 2.1, there exists $x \in \Omega$ and $(Sx)(t) = x(t)$ holds for all $t \in [t_0, \infty)_{\mathbb{T}}$.

Finally, by Lemma 2.2 and simple calculation, $x(t)$ is a bounded nonoscillatory solution of (1.1).□

Theorem 3.4 Suppose that $(H_1) \sim (H_4)$ hold, and τ is a monotone increasing mapping, and there exist $q_2 > q_1 > 1, s.t.\ q_1^2 > q_2, b(q_1^2 - q_2) > (q_2 - 1)(q_1 + q_2), q_1 \leq p(t) \leq q_2$, and for all $0 \leq u_i, v_i \leq b, i = 1, 2, \cdots, m$, $| f(t, v_1, v_2, \cdots, v_m) - f(t, u_1, u_2, \cdots, u_m) | \leq mf(t, b, b, \cdots, b) \sup_{1 \leq i \leq m} | v_i - u_i |$ holds. Then there exists a bounded nonoscillatory solution of (1.1).

Proof Let $\Omega = \{x : x \in BC[t_0, \infty)_{\mathbb{T}}, \frac{bq_2(q_1^2 - q_2)}{2q_1(q_2^2 - q_1)} \leq x(t) \leq b, \forall t \in [t_0, \infty)_{\mathbb{T}}\}$. Clearly, Ω is a nonempty closed subset in $BC[t_0, \infty)_{\mathbb{T}}$. From $(H_1) \sim (H_4)$, there exists sufficient large $T_0 \in [t_0, \infty)_{\mathbb{T}}$, when $t \in [T_0, \infty)_{\mathbb{T}}$,

$$\tau(t) \geq t_0, \tau^{-1}(t) \geq t_0, \delta_i(t) \geq t_0, i = 1, 2, \cdots, m, \tag{3.10}$$

$$| G(t) - G_0 | \leq \frac{1}{2(q_1 + q_2)}[(q_1^2 - q_2)b - (q_1 - 1)(q_1 + q_2)], \tag{3.11}$$

$$\frac{1}{(n-1)!} \int_{\tau^{-1}(T_0)}^{\infty} (\sigma(s) - t_0)^{n-1} f(s, b, \cdots, b) \Delta s \leq \frac{q_1 - 1}{2m}. \tag{3.12}$$

Define the operator $S : \Omega \to BC[t_0, \infty)_{\mathbb{T}}$ as follows

$$(Sx)(t) = \begin{cases} \frac{1}{p(\tau^{-1}(t))}\{\frac{bq_2}{q_1+q_2}[1 + q_1 + \frac{(q_1^2 - q_2)(q_2 + 1)}{2(q_2^2 - q_1)}] - x(\tau^{-1}(t)) + \\ \frac{(-1)^{n-1}}{(n-1)!} \int_{\tau^{-1}(t)}^{\infty} (\sigma(s) - \tau^{-1}(t))^{n-1} f(s, x(\delta_1(s)), \cdots, x(\delta_m(s))) \Delta s + G(\tau^{-1}(t)) - G_0\}, t \in [T_0, \infty)_{\mathbb{T}}; \\ (Sx)(T_0), t \in [t_0, T_0)_{\mathbb{T}}. \end{cases}$$

Firstly, we prove $S\Omega \subseteq \Omega$. From (H_4) and $(3.10) \sim (3.12)$, for $t \in [T_0, \infty)_{\mathbb{T}}, x \in \Omega$,

$$\begin{aligned}(Sx)(t) &\geq \frac{1}{q_2} \frac{bq_2}{q_1 + q_2}[1 + q_1 + \frac{(q_1^2 - q_2)(q_2 + 1)}{2(q_2^2 - q_1)}] - \frac{b}{q_1} \\ &\quad - \frac{1}{p(\tau^{-1}(t))} \frac{1}{(n-1)!} \int_{\tau^{-1}(T_0)}^{\infty} (\sigma(s) - t_0)^{n-1} f(s, b, \cdots, b) \Delta s - \frac{1}{p(\tau^{-1}(t))} | G(t) - G_0 | \\ &\geq \frac{b}{q_1 + q_2}[1 + q_1 + \frac{(q_1^2 - q_2)(q_2 + 1)}{2(q_2^2 - q_1)}] - \frac{b}{q_1} - \frac{q_1 - 1}{2q_1} - \frac{1}{2q_1(q_1 + q_2)}[(q_1^2 - q_2)b - (q_1 - 1)(q_1 + q_2)] \\ &= \frac{bq_2(q_1^2 - q_2)}{2q_1(q_2^2 - q_1)},\end{aligned}$$

$$\begin{aligned}(Sx)(t) &\leq \frac{1}{q_1} \frac{bq_2}{q_1 + q_2}[1 + q_1 + \frac{(q_1^2 - q_2)(q_2 + 1)}{2(q_2^2 - q_1)}] - \frac{bq_2(q_1^2 - q_2)}{2q_1(q_2^2 - q_1)} q_2 \\ &\quad + \frac{1}{p(\tau^{-1}(t))} \frac{1}{(n-1)!} \int_{\tau^{-1}(T_0)}^{\infty} (\sigma(s) - t_0)^{n-1} f(s, b, \cdots, b) \Delta s + \frac{1}{p(\tau^{-1}(t))} | G(t) - G_0 | \\ &\leq \frac{bq_2}{q_1(q_1 + q_2)}[1 + q_1 + \frac{(q_1^2 - q_2)(q_2 + 1)}{2(q_2^2 - q_1)}] - \frac{b(q_1^2 - q_2)}{2q_1(q_2^2 - q_1)} + \frac{q_1 - 1}{2q_1}\end{aligned}$$

$$+\frac{1}{2q_1(q_1+q_2)}[(q_1^2-q_2)b-(q_1-1)(q_1+q_2)]=b.$$

Therefore

$$\frac{bq_2(q_1^2-q_2)}{2q_1(q_2^2-q_1)}\le x(t)\le b.$$

So $S\Omega\subseteq\Omega$.

Secondly, we prove S is a contraction mapping on Ω. For $t\in[T_0,\infty)_{\mathbb{T}}, x,y\in\Omega$,from (H_4) and $(3.10),(3.12)$, we have

$$\begin{aligned}|(Sx)(t)-(Sy)(t)|&\le\frac{1}{p(\tau^{-1}(t))}[\,|x(\tau^{-1}(t))-y(\tau^{-1}(t))|+\frac{1}{(n-1)!}\int_{\tau^{-1}(T_0)}^{\infty}(\sigma(s)-\tau^{-1}(t))^{n-1}\\&\quad|f(s,x(\delta_1(s)),\cdots,x(\delta_m(s)))-f(s,y(\delta_1(s)),\cdots,y(\delta_m(s)))|\,\Delta s\,]\\&\le\frac{1}{q_1}\parallel x-y\parallel+\frac{q_1-1}{2q_1}\parallel x-y\parallel=\frac{q_1+1}{2q_1}\parallel x-y\parallel.\end{aligned}$$

Thus $\parallel Sx-Sy\parallel\le\frac{q_1+1}{2q_1}\parallel x-y\parallel$. Therefore S is a contraction mapping on Ω since $q_1>1$.

From Lemma 2.1, there exists $x\in\Omega$ and $(Sx)(t)=x(t)$ holds for all $t\in[t_0,\infty)_{\mathbb{T}}$.

Finally, by Lemma 2.2 and simple calculation, $x(t)$ is a bounded nonoscillatory solution of (1.1).□

As an application of Theorem $3.1\sim$ Theorem 3.4, we can obtain several relevant results on the following special time scale.

When $\mathbb{T}=R$, the equation (1.1) is the following differential equation

$$[x(t)+p(t)x(\tau(t))]^{(n)}+f[t,x(\delta_1(t)),x(\delta_2(t)),\cdots,x(\delta_m(t))]=g(t),\ t\in[t_0,\infty)_{\mathbb{T}},\tag{3.13}$$

where $t_0\in R, p,g,\tau,\delta_i\in C_{rd}([t_0,\infty),R), i=1,2,\cdots,m, f\in C([t_0,\infty)\times R^m,R)$.

The assumptions $(H_1)\sim(H_4)$ are the following $(H_1^{'})\sim(H_4^{'})$

$(H_1^{'})$ $\delta_i(t)\le t,\forall t\in[t_0,\infty)$, and $\lim\limits_{t\to\infty}\delta_i(t)=\infty, i=1,2,\cdots,m$;

$(H_2^{'})$ $\tau(t)\le t,\forall t\in[t_0,\infty)$, and $\lim\limits_{t\to\infty}\tau(t)=\infty$;

$(H_3^{'})$ $\exists G\in C^n([t_0,\infty),R), G_0\in R, s.t.\ G^{(n)}(t)=g(t),\forall t\in[t_0,\infty), \lim\limits_{t\to\infty}G(t)=G_0$;

$(H_4^{'})$ $x_if(t,x_1,x_2,\cdots,x_m)\ \ge\ 0, f(t,x_1,x_2,\cdots,x_m)$ is not decreasing for each $x_i\in R$, $i=1,2,\cdots,m,t\in[t_0,\infty)$, and $\exists b>2, s.t.\ \int_{t_0}^{\infty}(s-t_0)^{n-1}f(s,b,\cdots,b)ds<\infty$.

In Theorem $3.1\sim$ Theorem 3.4, the conditons $(H_1)\sim(H_4)$ are replaced by $(H_1^{'})\sim(H_4^{'})$ and other conditions are not changed, then there exists a bounded nonoscillatory solution of (3.13).

4 Applications

We consider the fifth-order forced neutral dynamic equation with time delay on $\mathbb{T}=[2,\infty)$

$$[x(t)+(\frac{1}{t}+\frac{1}{3})x(t-1)]^{(5)}+[\frac{119(t-1)}{t^7}+\frac{160}{t(t-1)^5}]x(t-1)=-\frac{1}{t^6},\tag{4.1}$$

where $m=1,n=5,p(t)=\frac{1}{t}+\frac{1}{3},\tau(t)=\delta_1(t)=t-1,g(t)=-\frac{1}{t^6},t_0=2,$

$$f(t,x(\delta_1(t)))=[\frac{119(t-1)}{t^7}+\frac{160}{t(t-1)^5}]x(t-1).$$

Let $b>2$, we have

$$(s-2)^4f(s,b)=(s-2)^4[\frac{119(s-1)}{s^7}+\frac{160}{s(s-1)^5}]b<\frac{119b}{s^2}+\frac{160b}{(s-1)^2},$$

and

$$\int_2^\infty (\frac{119b}{s^2} + \frac{160b}{(s-1)^2})ds < \infty.$$

Therefore the conditions of Theorem 3.1 are satisfied and there exists a bounded nonoscillatory solution of (4.1). In fact, $x(t) = \frac{1}{t} + 1$ is a nonoscillatory solution of (4.1).

5 Conclusions

In this paper, we obtain several existence theorems of nonoscillatory solutions of a class of higher order forced neutral dynamic equations with time delay on time scales and give an example to illustrate the applications of the obtained results.

Competing Interests

The authors declare that no competing interests exist.

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