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Positive Radial Solutions of the p-Laplacian in an Annulus with a Superlinear Nonlinearity with Zeros

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Abstract

In this paper, we study existence, and the asymptotic behavior, with respect to λ, of positive radially symmetric solutions of

$$-\triangle_p u = \lambda h(x, u)$$

in annular domains in $\mathbb{R}^N, N \geq 2$. The nonlinear term has a superlinear local growth at infinity, and is nonnegative.

Keywords: Positive radial solutions; p-Laplacian; superlinear; asymptotic behavior.

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1 Introduction and Main Results

We study existence and the asymptotic behavior with respect to λ of positive solutions of the problem

$$\begin{cases} -\triangle_p u = \lambda h(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \qquad (A_\lambda)$$

where $\Omega = \{x \in \mathbb{R}^N : r_1 < |x| < r_2\}$ with $0 < r_1 < r_2$, $\triangle_p u = \text{div}(|\nabla u|^{p-2}\nabla u)$ $(1 < p \leq N)$ is the p-laplacian, and where $\lambda > 0$ is a real parameter and h is a radial nonnegative nonlinearity which is locally superlinear at $+\infty$.

In recent years, the existence, non-existence, asymptotic behavior and uniqueness of the positive solutions for the following quasilinear eigenvalue problems

$$\begin{cases} -\triangle_p u = \lambda f(u(x)), & x \in \Omega; \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \qquad (B_\lambda)$$

have been studied by many authors. In [1] Maya and Petr proved nonexistence results for when Ω is a unit ball in $\mathbb{R}^N$ and f has only one zero. In [2], nonexistence and existence results are proved when Ω is a connected and bounded subset of $\mathbb{R}^N$. [3] studied elliptic systems related to (B_λ) and proved the existence of positive solutions to (B_λ) in some sublinear cases. The result of nontrivial solutions for p-Laplacian systems be proved by [4]. When f is strictly increasing on $\mathbb{R}^+, f(0) = 0, \lim\limits_{s\to 0^+} f(s)/s^{p-1} = 0$ and $f(s) \leq \alpha_1 + \alpha_2 s^\mu, 0 < \mu < p-1, \alpha_1, \alpha_2 > 0$, it was shown in [5] that there exist at least two positive solutions for Eqs (B_λ) when λ is sufficiently large. If $\lim\limits_{s\to 0^+} \inf f(s)/s^{p-1} > 0, f(0) = 0$ and the monotonicity hypothesis $(f(s)/s^{p-1})' < 0$ holds for all $s > 0$, it was proved i n[6] that the problem (B_λ) has a unique positive solution when λ is sufficiently large. Moreover, it was also shown in [7] that problem (B_λ) has a unique positive large solution and at least one positive small solution when λ is large if f is nondecreasing; there exist $\alpha_1, \alpha_2 > 0$ such that $f(s) \leq \alpha_1 + \alpha_2 s^\beta, 0 < \beta < p-1$; $\lim\limits_{s\to 0^+} \dfrac{f(s)}{s^{p-1}} = 0$, and there exist $T, Y > 0$ with $Y \geq T$ such that

$$(f(s)/s^{p-1})' > 0 \text{ for } s \in (0,T)$$

and

$$(f(s)/s^{p-1})' < 0 \text{ for } s > Y.$$

Recently, [8] considered the case when Ω is an annular domains, and obtained the existence of positive large solutions for the problem (B_λ) when λ sufficiently small. [9] proved the singular problem (B_λ) has a unique positive radial solution if f is a continuous function and positive on $\overline{\Omega} = B_R$ (here B_R is a ball). The existence of entire solutions for (B_λ) with singular and non-singular has been considered in [10].

When $p = 2, f(0) < 0$, Ω is an annulus or a ball and f has more than one zero, the related results for the problem (B_λ) have been obtained by [11] and [12]. In [13,14], when $p = 2, f(0) < 0$, f is a monotone nondecreasing nonlinearity and has only one zero, this problem has been studied by [13] in the ball, and by [14] in the annulus. The asymptotic behavior of positive solution have been obtained by [15,16].

When $p = 2$, the existence, multiplicity, and the behavior of positive radially solutions of (A_λ) has been studied extensively, see for example [17] and the reference therein. When Ω is a bounded domain of $\mathbb{R}^N$, (A_λ) is studied in [18] by the same authors as [17].

Motivated by the results of the above cited papers, we shall attempt to treat such equation (A_λ), the results of the literature [17] are extended to the problem (A_λ).

In fact, in order to study the solution of (A_λ), one can make a standard change of variables. In the case $N \geq p+1$, if $t = -\frac{A}{r^{(N-p)/(p-1)}} + B$ and $v(t) = u(r)$, where

$$A = \frac{(r_1 r_2)^{\frac{N-p}{p-1}}}{r_2^{\frac{N-p}{p-1}} - r_1^{\frac{N-p}{p-1}}} \text{ and } B = \frac{r_2^{\frac{N-p}{p-1}}}{r_2^{\frac{N-p}{p-1}} - r_1^{\frac{N-p}{p-1}}},$$

then Problem (A_λ) transforms into the boundary value problem for the ODE

$$\begin{cases} (|v'(t)|^{p-2}v'(t))' + \lambda q(t) f(t, v(t)) = 0, & t \in (0,1), \\ v(0) = v(1) = 0, \end{cases} \tag{P_λ}$$

where

$$f(t,v) = h((\frac{A}{B-t})^{\frac{p-1}{N-p}}, v(t)) \text{ and } q(t) = (\frac{N-p}{p-1})^{1-p} \frac{A^{\frac{(p-1)^2}{N-p}}}{(B-t)^{\frac{(p-1)(N-1)}{N-p}}}.$$

In the case $N = p$, one sets $r = r_2(\frac{r_1}{r_2})^t$ and $v(t) = u(r)$, obtaining again the Problem (P_λ), this time with

$$f(t,v) = h(r_2(\frac{r_1}{r_2})^t, v(t)) \text{ and } q(t) = [r_2(\frac{r_1}{r_2})^t(\ln\frac{r_2}{r_1})^{-1}]^{p-1}.$$

Note that, in both cases, the function $q(t)$ is well defined, continuous and bounded between positive constants in the interval $[0,1]$.

For our purpose, we shall restrict our attention to the ordinary boundary value problem (P_λ), where the function $q(t)$ is continuous and positive on the interval $[0,1]$, while for f we consider the following assumptions:

(H_1) The function $f : [0,1]\times[0,+\infty) \to [0,+\infty)$ is continuous, $f(t,0) = 0$ and $f(t,v) > 0$ if $v > 0$.

(H_2) There exists a continuous function $b : [0,1] \to (0,+\infty)$ such that

$\lim_{v\to 0^+} \frac{f(t,v)}{v^{p-1}} = b(t)$ uniformly in $t \in [0,1]$.

(H_3) There exist positive constants $0 < \alpha < \beta < 1$ such that

$\lim_{v\to +\infty} \frac{f(t,v)}{v^{p-1}} = +\infty$ uniformly in $t \in [\alpha,\beta]$.

Before stating our results, we need to introduce some notations. Let $m : [0,1] \to \mathbb{R}$ be a continuous function and consider the eigenvalue problem

$$\begin{cases} -(|v'(t)|^{p-2}v'(t))' = \lambda m(t)|v(t)|^{p-2}v(t), & in\ (0,1), \\ v(0) = v(1) = 0. \end{cases} \tag{1.1}$$

In particular $\lambda_{1,m} > 0$ is called the first eigenvalue of (1.1) and the associated eigenfunction will be denoted by $\phi_{1,m}$. It is known that $\phi_{1,m} > 0$. Then, $\lambda_{1,m}$ satisfies

$$\begin{cases} -(|\phi'_{1,m}(t)|^{p-2}\phi'_{1,m}(t))' = \lambda_{1,m} m(t)\phi_{1,m}^{p-1}(t), & in\ (0,1), \\ \phi_{1,m}(0) = \phi_{1,m}(1) = 0. \end{cases} \tag{1.2}$$

In addition, we have

$$\int_0^1 |v'(t)|^p dt \geq \lambda_{1,m} \int_0^1 m(t)|v(t)|^p dt \text{ for any } v \in H_0^1(0,1), \tag{1.3}$$

where the equality holds if and only if v is a multiple of $\phi_{1,m}$.

Since $q(t)$ and $b(t)$ are continuous and positive in $[0,1]$, there exist $C_i (i = 1,2,3,4) > 0$ such that

$$\max_{t\in[0,1]} q(t) = C_1, \quad \max_{t\in[0,1]} b(t) = C_2,$$
$$\min_{t\in[0,1]} q(t) = C_3, \quad \min_{t\in[0,1]} b(t) = C_4.$$

The main results in this paper are the following theorems.

Theorem 1.1. Suppose $f(t,v)$ satisfies the hypotheses $(H_1) \sim (H_3)$. Then there exists a positive solution of Problem (P_λ), for every $0 < \lambda < \lambda^*$, where $\lambda^* = \frac{1}{(1+\varepsilon)C_1C_2}$.

Theorem 1.2. Suppose $f(t,v)$ satisfies the hypotheses $(H_1) \sim (H_3)$ and $\{v_\lambda\}$ is a family of positive solutions of Problem (P_λ). Then one has

$$\|v_\lambda\|_\infty \to +\infty \text{ when } \lambda \to 0^+.$$

The paper is organized as follows. In Section 2, we establish some notations, as well as some basic facts, and we state some known results that will be used in the paper. Section 3 contains the proof of the solution for λ small (Theorem 1.1), a Krasnoselskii fixed point theorem is used. Finally, in Section 4, we study the asymptotic behavior of the solutions (Theorem 1.2).

2 Preliminaries

We consider the Banach space $X = C([0,1])$ endowed with the norm $\|v\|_\infty = \max_{t\in[0,1]} |v(t)|$. Set

$$\Phi_p(s) = |s|^{p-2}s = \begin{cases} s^{p-1}, s \geq 0, \\ -(-s)^{p-1}, s < 0, \end{cases}$$

then

$$\Phi_p^{-1}(s) = \begin{cases} s^{\frac{1}{p-1}}, s \geq 0, \\ -(-s)^{\frac{1}{p-1}}, s < 0. \end{cases}$$

Let v be a solution of (P_λ), then

$$v(t) = \begin{cases} \int_0^t \Phi_p^{-1}(\lambda \int_s^\sigma q(\tau)f(\tau, v(\tau))d\tau)ds, & 0 \leq t \leq \sigma, \\ -\int_t^1 \Phi_p^{-1}(-\lambda \int_\sigma^s q(\tau)f(\tau, v(\tau))d\tau)ds, & \sigma \leq t \leq 1, \end{cases} \tag{2.1}$$

where $\sigma \in (0,1)$ is the unique solution of the equation $\Theta v(t) = 0$, $0 \leq t \leq 1$, where the map $\Theta : X \to X$ is defined by

$$\Theta v(t) = \int_0^t \Phi_p^{-1}(\lambda \int_s^t q(\tau)f(\tau, v(\tau))d\tau)ds + \int_t^1 \Phi_p^{-1}(-\lambda \int_t^s q(\tau)f(\tau, v(\tau))d\tau)ds. \tag{2.2}$$

Conversely, if v is a function satisfying (2.1), then v is a solution of (P_λ).

It is easy to see if $v(t)$ is a nonnegative solution of Problem (P_λ), then it is a concave function. In view of this fact, we define Q be a cone in X by

$$Q = \{v \in X : v \text{ is concave and } v(0) = v(1) = 0\}.$$

Let $T : Q \to X$ be defined by

$$Tv(t) = \begin{cases} \int_0^t \Phi_p^{-1}(\lambda \int_s^\sigma q(\tau)f(\tau, v(\tau))d\tau)ds, & 0 \le t \le \sigma, \\ -\int_t^1 \Phi_p^{-1}(-\lambda \int_\sigma^s q(\tau)f(\tau, v(\tau))d\tau)ds, & \sigma \le t \le 1. \end{cases}$$

By virtue of Lemma 2.1, the operator T is well defined. Then its nontrivial fixed points in Q correspond to the positive solutions of Problem (P_λ).

Lemma 2.1. Assume (H_1) holds. Then for any $v \in Q$, $\Theta v(t) = 0$ has a unique solution in $(0, 1)$.

Proof. Let $\alpha(\tau) = \lambda q(\tau)f(\tau, v(\tau))$, then $\alpha(\tau) > 0$ for $\tau \in (0, 1)$. Therefore, $\Theta v(0) < 0$ and $\Theta v(1) > 0$. It follows from the continuity of $\Theta v(t)$ that $\Theta v(t) = 0$ has at least one solution in $(0, 1)$. Moreover, it is not difficult to check that $\Theta v(t)$ is nondecreasing function on $[0, 1]$.

If σ_1, $\sigma_2 \in (0, 1)$ are two different solutions of $\Theta v(t) = 0$, and without loss of generality, we assume that $\sigma_1 < \sigma_2$. We consider

$$\begin{aligned} \int_{\sigma_1}^{\sigma_2} \Phi_p^{-1}(\int_s^{\sigma_2} \alpha(\tau)d\tau)ds &= \int_0^{\sigma_2} \Phi_p^{-1}(\int_s^{\sigma_2} \alpha(\tau)d\tau)ds - \int_0^{\sigma_1} \Phi_p^{-1}(\int_s^{\sigma_2} \alpha(\tau)d\tau)ds \\ &\le \int_0^{\sigma_2} \Phi_p^{-1}(\int_s^{\sigma_2} \alpha(\tau)d\tau)ds - \int_0^{\sigma_1} \Phi_p^{-1}(\int_s^{\sigma_1} \alpha(\tau)d\tau)ds. \end{aligned}$$

Now, because of $\Theta v(\sigma_1) = \Theta v(\sigma_2) = 0$, we have

$$\begin{aligned} \int_{\sigma_1}^{\sigma_2} \Phi_p^{-1}(\int_s^{\sigma_2} \alpha(\tau)d\tau)ds &\le \int_{\sigma_1}^1 \Phi_p^{-1}(-\int_{\sigma_1}^s \alpha(\tau)d\tau)ds - \int_{\sigma_2}^1 \Phi_p^{-1}(-\int_{\sigma_2}^s \alpha(\tau)d\tau)ds \\ &\le \int_{\sigma_1}^{\sigma_2} \Phi_p^{-1}(-\int_{\sigma_1}^s \alpha(\tau)d\tau)ds < 0, \end{aligned}$$

which is contrary to $\int_{\sigma_1}^{\sigma_2} \Phi_p^{-1}(\int_s^{\sigma_2} \alpha(\tau)d\tau)ds > 0$. Therefore, $\sigma_1 = \sigma_2$, and $\Theta v(t) = 0$ has the unique solution.

Now we state the following well-known result without proof (compare [19]-[23]).

Lemma 2.2. Let X be a Banach space endowed with a norm $\|\cdot\|$, and let $Q \subset X$ be a cone in X. For $R > 0$, define $Q_R = \{v \in Q : \|v\| < R\}$. Let r and R be numbers satisfying $0 < r < R$. Assume that $T : \overline{Q}_R \to Q$ is a completely continuous operator such that

$$\|Tv\| < \|v\|, \text{for all } v \in \partial Q_r \text{ and } \|Tv\| > \|v\|, \text{for all } v \in \partial Q_R, \text{or}$$
$$\|Tv\| > \|v\|, \text{for all } v \in \partial Q_r \text{ and } \|Tv\| < \|v\|, \text{for all } v \in \partial Q_R,$$

where $\partial Q_R = \{v \in Q : \|v\| = R\}$. Then T has a fixed point $v \in Q$, with $r < \|v\| < R$.

We also state some elementary properties of concave functions.

Lemma 2.3. Given a function v in the cone Q and a point $p \in (0, 1)$, the following estimates hold:

$$(i)\ v(t) \ge \begin{cases} \frac{t}{p}v(p), & t < p, \\ \frac{1-t}{1-p}v(p), & t > p, \end{cases} \quad \text{and} \quad (ii)\ v(t) \le \begin{cases} \frac{t}{p}v(p), & t > p, \\ \frac{1-t}{1-p}v(p), & t < p. \end{cases}$$

Moreover, for all $0 < t_0 < t_1 < 1$, we have

$$(iii)\ \min_{t \in [t_0, t_1]} v(t) \ge c_{t_0,t_1}\|v\|_\infty,$$

where $c_{t_0,t_1} := \min\{t_0, 1 - t_1\}$.

Proof. The proof is same as [17].

3 Proof of Theorem 1.1.

In this section we will show the existence of a solution for $\lambda \in (0, \lambda^*)$, by verifying, in the next three lemmas, the hypotheses of Lemma 2.2.

Lemma 3.1. $T : \overline{Q}_R \to Q$ is a completely continuous operator for $\lambda \in (0, \lambda^*)$.

Proof. It is obvious that $T(\overline{Q}_R) \subset Q$. We now show that T is compact. Let $\{v_n\}_{n \in N}$ be a bounded sequence in Q and let $R > 0$ be such that $\|v_n\| \le R$ for all $n \in N$. Hence by the definition of T, we have

$$(Tv_n)'(t) = \begin{cases} \Phi_p^{-1}(\lambda \int_t^{\sigma} q(\tau) f(\tau, v_n(\tau)) d\tau) ds, & 0 \le t \le \sigma, \\ \Phi_p^{-1}(-\lambda \int_{\sigma}^{t} q(\tau) f(\tau, v_n(\tau)) d\tau) ds, & \sigma \le t \le 1. \end{cases}$$

Then it is easy to see that $\{Tv_n\}_{n \in N}$ and $\{(Tv_n)'\}_{n \in N}$ are uniformly bounded sequences. It follows from the Ascoli-Arzela theorem that T is relatively compact.

It remains to show the continuity of T. Let $v_n, v \in \overline{Q}_R$, and $v_n \to v$ uniformly on $[0, 1]$. We need to show that $Tv_n \to Tv$ uniformly on $[0, 1]$. We have

$$\begin{aligned} &Tv_n - Tv \\ &= \begin{cases} \int_0^t [\Phi_p^{-1}(\lambda \int_s^{\sigma} q(\tau) f(\tau, v_n(\tau)) d\tau) ds - \Phi_p^{-1}(\lambda \int_s^{\sigma} q(\tau) f(\tau, v(\tau)) d\tau)] ds, & 0 \le t \le \sigma, \\ \int_t^1 [\Phi_p^{-1}(-\lambda \int_{\sigma}^{s} q(\tau) f(\tau, v(\tau)) d\tau) ds - \Phi_p^{-1}(-\lambda \int_{\sigma}^{s} q(\tau) f(\tau, v_n(\tau)) d\tau)] ds, & \sigma \le t \le 1. \end{cases} \end{aligned}$$

Pay attention to Φ_p^{-1} is continuous, then $T : \overline{Q}_R \to Q$ is continuous. The proof is completed.

Lemma 3.2. Suppose conditions (H_1) and (H_3) hold. Let $\| \cdot \|$ be a norm on $C([0, 1])$ which is equivalent to $\| \cdot \|_\infty$. Then, for all $\Lambda, K > 0$ there exists $R > 0$ such that, for all $\lambda \ge \Lambda$ and all $v \in \{v \in Q : \|v\| \ge R\}$, we have

$$\|Tv\| > K\|v\|.$$

Proof. Without loss of generality we will give the proof using the norm $\| \cdot \|_\infty$. By hypothesis (H_3), given $M > 0$, there exists $N > 0$ such that $v > N$ implies $f(s, v) \ge Mv^{p-1}$, for all $s \in [\alpha, \beta]$. By estimate (iii) in Lemma 2.3, $v(s) \ge c_{\alpha,\beta}\|v\|_\infty$ for $s \in [\alpha, \beta]$. Then, if we choose $\|v\|_\infty \ge R > \frac{N}{c_{\alpha,\beta}}$, we have

$$f(s, v) \ge M(c_{\alpha,\beta}\|v\|_\infty)^{p-1}.$$

Note, from the definition of $Tv(t)$, that $Tv(\sigma)$ is the maximum value of $Tv(t)$ on $[0, 1]$. If $\sigma \in [\frac{1}{4}, \frac{3}{4}]$, we have

$$f(s, v) \ge M(\frac{1}{4}\|v\|_\infty)^{p-1}.$$

So,

$$\begin{aligned}
&2\|Tv\|_\infty \\
\geq& \int_{\frac{1}{4}}^{\sigma} \Phi_p^{-1}(\lambda \int_s^{\sigma} q(\tau) f(\tau, v(\tau)) d\tau) ds - \int_{\sigma}^{\frac{3}{4}} \Phi_p^{-1}(-\lambda \int_{\sigma}^{s} q(\tau) f(\tau, v(\tau)) d\tau) ds \\
=& \int_{\frac{1}{4}}^{\sigma} (\lambda \int_s^{\sigma} q(\tau) f(\tau, v(\tau)) d\tau)^{\frac{1}{p-1}} ds + \int_{\sigma}^{\frac{3}{4}} (\lambda \int_{\sigma}^{s} q(\tau) f(\tau, v(\tau)) d\tau)^{\frac{1}{p-1}} ds \\
\geq& \int_{\frac{1}{4}}^{\sigma} (\Lambda \int_s^{\sigma} C_3 M(\frac{1}{4}\|v\|_\infty)^{p-1} d\tau)^{\frac{1}{p-1}} ds + \int_{\sigma}^{\frac{3}{4}} (\Lambda \int_{\sigma}^{s} C_3 M(\frac{1}{4}\|v\|_\infty)^{p-1} d\tau)^{\frac{1}{p-1}} ds \\
=& \frac{1}{4}\|v\|_\infty (\Lambda C_3 M)^{\frac{1}{p-1}} \int_{\frac{1}{4}}^{\sigma} (\sigma - s)^{\frac{1}{p-1}} ds + \frac{1}{4}\|v\|_\infty (\Lambda C_3 M)^{\frac{1}{p-1}} \int_{\sigma}^{\frac{3}{4}} (s-\sigma)^{\frac{1}{p-1}} ds \\
=& \frac{p-1}{4p} (\Lambda C_3 M)^{\frac{1}{p-1}} [(\sigma - \frac{1}{4})^{\frac{p}{p-1}} + (\frac{3}{4} - \sigma)^{\frac{p}{p-1}}] \|v\|_\infty,
\end{aligned}$$

and

$$\|Tv\|_\infty \geq \frac{p-1}{8p} (\Lambda C_3 M)^{\frac{1}{p-1}} [(\sigma - \frac{1}{4})^{\frac{p}{p-1}} + (\frac{3}{4} - \sigma)^{\frac{p}{p-1}}] \|v\|_\infty.$$

By setting $M > (\frac{8p}{p-1}K)^{p-1}(\Lambda C_3)^{-1}[(\sigma - \frac{1}{4})^{\frac{p}{p-1}} + (\frac{3}{4} - \sigma)^{\frac{p}{p-1}}]^{1-p}$, one obtains $\|Tv\|_\infty > K\|v\|_\infty$.

For $\sigma \in (\frac{3}{4}, 1)$, it is easy to see

$$\|Tv\|_\infty \geq \int_{\frac{1}{4}}^{\frac{3}{4}} \Phi_p^{-1}(\lambda \int_s^{\frac{3}{4}} q(\tau) f(\tau, v(\tau)) d\tau) ds.$$

On the other hand, we have

$$\|Tv\|_\infty \geq -\int_{\frac{1}{4}}^{\frac{3}{4}} \Phi_p^{-1}(-\lambda \int_{\frac{1}{4}}^{s} q(\tau) f(\tau, v(\tau)) d\tau) ds,$$

if $\sigma \in (0, \frac{1}{4})$. Therefore, similar arguments show that $\|Tv\|_\infty > K\|v\|_\infty$.

Lemma 3.3. Suppose conditions (H_1) and (H_2) hold. Then, for all $\lambda \in (0, \lambda^*)$, there exists a norm $\|\cdot\|_*$ equivalent to $\|\cdot\|_\infty$ and $r > 0$ such that, for all $v \in \{v \in Q : \|v\|_* = r\}$, we have

$$\|Tv\|_* < \|v\|_*.$$

Proof. We consider the norm

$$\|v\|_* = |v|_E = \inf\{\xi : \xi(\|\phi_{1,qb}\|_\infty + E) \geq v\} = \frac{\|v\|_\infty}{\|\phi_{1,qb}\|_\infty + E},$$

which is equivalent to $\|\cdot\|_\infty$.

By hypotheses (H_1) and (H_2), there exists a $\delta = \delta(\varepsilon) > 0$ such that $0 < v < \delta$ implies $f(s, v) < (1+\varepsilon)b(s)v^{p-1}$ for all $s \in [0, 1]$. Let $r > 0$ be such that $r(\|\phi_{1,qb}\|_\infty + E) < \delta$, so that $|v|_E = r$ implies $\|v\|_\infty < \delta$.

If $v \in Q$ with $|v|_E = r$, we have when $0 \le t \le \sigma$,

$$\begin{aligned}
Tv(t) &= \int_0^t (\lambda \int_s^\sigma q(\tau) f(\tau, v(\tau)) d\tau)^{\frac{1}{p-1}} ds \\
&< \int_0^t (\lambda^* \int_s^\sigma (1+\varepsilon) q(\tau) b(\tau) v^{p-1} d\tau)^{\frac{1}{p-1}} ds \\
&= ((1+\varepsilon)\lambda^*)^{\frac{1}{p-1}} \int_0^t (\int_s^\sigma q(\tau) b(\tau) \frac{v^{p-1}}{(\|\phi_{1,qb}\|_\infty + E)^{p-1}} (\|\phi_{1,qb}\|_\infty + E)^{p-1} d\tau)^{\frac{1}{p-1}} ds \\
&\le ((1+\varepsilon) C_1 C_2 \lambda^*)^{\frac{1}{p-1}} (\|\phi_{1,qb}\|_\infty + E) |v|_E \int_0^t (\sigma - s)^{\frac{1}{p-1}} ds \\
&= \frac{p-1}{p} [\sigma^{\frac{p}{p-1}} - (\sigma - t)^{\frac{p}{p-1}}] (\|\phi_{1,qb}\|_\infty + E) |v|_E \\
&< (\|\phi_{1,qb}\|_\infty + E) |v|_E .
\end{aligned}$$

Therefore $|Tv|_E < |v|_E$. It is similar for $\sigma \le t \le 1$. Thus, $|Tv|_E < |v|_E$ for all $t \in [0, 1]$.

Proof of Theorem 1.1. By Lemma 3.2 and 3.3, there exists $\Lambda \in (0, \lambda^*)$ such that the above conclusions hold. So combined with Lemma 3.1, the existence of the positive solution is a consequence of Lemma 2.2 (using the equivalent norm $\|\cdot\| = |\cdot|_E$ from Lemma 3.3).

4 Proof of Theorem 1.2

In this section we will study the asymptotic behavior of the positive solution of Problem (P_λ) with respect to the parameter λ sufficiently small.

Proof of Theorem 1.2. Suppose by contradiction that there exists a sequence $\lambda_n \to 0^+$ and a constant $l > 0$ such that $\|v_{\lambda_n}\|_\infty \le l$. By the continuity of f and (H_2), there exists a positive constant C such that $f(t, v) < Cv^{p-1}$ for $0 \le v \le l$. Then, when $0 \le t \le \sigma$,

$$\begin{aligned}
v_{\lambda_n}(t) &= \int_0^t (\lambda_n \int_s^\sigma q(\tau) f(\tau, v_{\lambda_n}(\tau)) d\tau)^{\frac{1}{p-1}} ds \\
&\le (\lambda_n C)^{\frac{1}{p-1}} \int_0^t (\int_s^\sigma q(\tau) v_{\lambda_n}^{p-1}(\tau) d\tau)^{\frac{1}{p-1}} ds \\
&\le (\lambda_n C C_1)^{\frac{1}{p-1}} \|v_{\lambda_n}\|_\infty \int_0^t (\sigma - s)^{\frac{1}{p-1}} ds \\
&\le \frac{p-1}{p} (\lambda_n C C_1)^{\frac{1}{p-1}} \sigma^{\frac{p}{p-1}} \|v_{\lambda_n}\|_\infty ,
\end{aligned}$$

therefore,

$$1 \le \frac{p-1}{p} (\lambda_n C C_1)^{\frac{1}{p-1}} \sigma^{\frac{p}{p-1}},$$

but this is impossible, since $\lambda_n \to 0^+$. Similar arguments are for $\sigma \le t \le 1$. So $\|v_\lambda\|_\infty \to +\infty$ when $\lambda \to 0^+$.

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Competing Interests

The authors declare that no competing interests exist.

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