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The Single-valued Character of Some Class of Multi-valued Admissible Mappings

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Abstract

In the countably dimensional, separable and locally compact spaces, some class of admissible mappings was defined that is also closed due to composition and that has similar properties to the properties of single-valued mappings. A certain property of these maps was applied in the proof of Theorem 3.7, which is the main result of this work.

Keywords: Absolute retracts, absolute neighborhood retracts, countable dimension, trivial shape, TSA-map, homotopy extension, TSA-contractible, elementary extension.

MSC 2010 classification: 54C60, 55P55, 54C20, 55M15, 55P05

1 Introduction

In 1976, L. Górniewicz defined the notion of multi-valued admissible mappings, the composition of which is also a multi-valued admissible mapping. It turns out that thanks to Theorem 2.8, some class of multi-valued, admissible mappings can be displayed in the class of admissible mappings, of a topological character, that is closed due to composition and has many properties similar to the properties of single-valued mappings. Particularly in Theorem 3.7 it has been proven that in Euclidean spaces closed sets of ANR-type that are mutli-valuedly contractible to a point are absolute retracts.

2 Preliminaries

Throughout this paper all topological spaces are assumed to be metrizable. A continuous mapping $f : X \to Y$ is called proper if for every compact set $K \subset Y$ the set $f^{-1}(K)$ is nonempty and compact. A proper map $p : X \to Y$ is called a cell-like map provided for every $y \in Y$ the set $p^{-1}(y)$ is of trivial shape. The notion of shape (Sh) is understood in the sense of Borsuk (see (1)). Let X and Y be two spaces and assume that for every $x \in X$ a non-empty and closed subset $\varphi(x)$ of Y is given. In such

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a case we say that $\varphi : X \multimap Y$ is a multi-valued mapping. For a multi-valued mapping $\varphi : X \multimap Y$ and a subset $U \subset Y$, we let:

$$\varphi^{-1}(U) = \{x \in X;\ \varphi(x) \subset U\}.$$

If for every open $U \subset Y$ the set $\varphi^{-1}(U)$ is open, then φ is called an upper semi-continuous mapping; we shall write that φ is u.s.c. A space X is of countable dimension if

$$X = \bigcup_{n=1}^{\infty} X_n, \text{ where } dim X_n < \infty \text{ for all } n. \tag{2.1}$$

Let $\varphi : X \multimap Y$ be a multi-valued map and let $A \subset X$ be a non-empty set. The symbol φ_A will be used to denote the mapping $\varphi_A : A \multimap Y$ given by the formula:

$$\varphi_A(x) = \varphi(x) \text{ for each } x \in A. \tag{2.2}$$

Definition 2.1 *We say that an admissible map (see (2)) $\varphi : X \multimap Y$ is TSA-type (we write $\varphi \in TSA$) if there exist a selected pair $(p, q) \subset \varphi$ and a metrizable space Z such that $p : Z \to X$ is a cell-like map and $q : Z \to Y$ is a continuous map.*

Definition 2.2 *We say that an s-admissible map (see (2)) $\varphi : X \multimap Y$ is $STSA$-type (we write $\varphi \in STSA$) if there exist a selected pair $(p, q) = \varphi$ and a metrizable space Z such that $p : Z \to X$ is a cell-like map and $q : Z \to Y$ is a continuous map.*

Proposition 2.3 (see (3; 4)) Let $\varphi : X \multimap Y$ be a TSA ($STSA$)-type, $f : P \to X$, $g : Y \to T$ continuous maps and $A \subset X$ a non-empty set. Then the following conditions are satisfied:
2.3.1 $(g \circ \varphi) \in TSA$ ($STSA$),
2.3.2 $(\varphi \circ f) \in TSA$ ($STSA$),
2.3.3 $(\varphi_A) \in TSA$ ($STSA$).
The following remark is obvious:

Remark 2.4 *An u.s.c. map $\varphi : X \multimap Y$ such that for each $x \in X$ $\varphi(x)$ is compact and of trivial shape, is $STSA$-type.*

Definition 2.5 (see (4)) *Let $\varphi, \psi : X \multimap Y$, $\varphi, \psi \in TSA$. We say that an admissible map $\chi : X \times [0, 1] \multimap Y$ is a* TSA-homotopy *connecting φ with ψ if the following conditions are satisfied:*
2.5.1 $\chi \in TSA$,
2.5.2 $\chi(x, 0) = \varphi(x)$ *and* $\chi(x, 1) = \psi(x)$ *for each* $x \in X$.
In this case the maps φ and ψ are said to be TSA-homotopic (notion: $\varphi \sim_{TSA} \psi$).

Definition 2.6 (see (4)) *Let X be a metrizable space. We say that a space X is* TSA -contractible *to a point $x_0 \in X$ (TSA-contractible) if there exists a TSA-homotopy $\chi : X \times [0, 1] \multimap X$ such that*

$$\chi(x, 0) = \{x\},\ \ \chi(x, 1) = \{x_0\} \text{ for each } x \in X.$$

Definition 2.7 (see (5)) *A proper surjection $f : X \to Y$ is called a shape equivalence if for every ANR Z the map f produces a one-to-one correspondence f^* between the homotopy classes of $C(Y, Z)$ and $C(X, Z)$. Here $C(A, B)$ stands for the space of continuous mappings from A into B.*
For this article, the most important piece of information is that if the mapping $f : X \to Y$ is a shape equivalence, then

$$ShX = ShY.$$

Theorem 2.8 (see (6)) Let X and Y be compact, metrizable spaces. *Let $f : X \to Y$ be a cell-like map such that the set*

$$\{y \in Y : f^{-1}(y) \text{ is a non-degenerate set}\}$$

is contained in a compact and countable dimensional subset of Y. Then for every closed subset A of Y the map

$$f_{/f^{-1}(A)} : f^{-1}(A) \to A$$

is a shape equivalence.

Definition 2.9 *Let $A \subset X, A \neq X$ be a non-empty set. We say that a multi-valued u.s.c. map $\varphi : A \multimap Y$ has an elementary extension if there exists an u.s.c. map $\widetilde{\varphi} : X \multimap Y$ such that $\widetilde{\varphi}_{X \setminus A} : X \setminus A \multimap Y$ is a continuous function and for each $x \in A$ $\widetilde{\varphi}(x) = \varphi(x)$.*

3 Main Result

If it is assumed that a metrizable space X is countably dimensional, then it shall be denoted by $X \in C_D$; if it is separable, then it will be denoted by $X \in S$; and if it is locally compact, the denotation is $X \in C_L$. If a metrizable space X satisfies a couple of the above conditions, e.g. it is both locally compact and separable, then it will be denoted by $X \in C_L S$. Firstly, the following fact should be noted: (Proposition 8.10, p. 40, see (2))

Proposition 3.1 *Let $p_1 : X \to Y$ and $p_2 : Y \to Z$ be cell-like maps. Assume that $Y \in C_D S$. Then*

$$p = p_2 \circ p_1 : X \to Z$$

is a cell-like map.

Proof. Let $z \in Z$. We observe that the map

$$p_1' : p_1^{-1}(p_2^{-1}(z)) \to p_2^{-1}(z), \ \ p_1'(x) = p_1(x) \ \text{ for each } \ x \in p_1^{-1}(p_2^{-1}(z))$$

is cell-like. Hence and from Theorem 2.8

$$Shp_1^{-1}(p_2^{-1}(z)) = Shp_2^{-1}(z),$$

so p is a cell-like map. □

The following class of multi-valued mappings will be defined:

Definition 3.2 We say that a TSA- map $\varphi : X \multimap Y$ is TSA_{C_D}-type (we write $\varphi \in$TSA$_{C_D}$) if there exist a pair $(p, q) \subset \varphi$ such that $p : \Gamma \to X$ is a cell-like map and $\Gamma \in C_D S$.

From the Proposition 3.1 it results that the set of multi-valued mappings of TSA_{C_D}-type is closed due to composition.

Proposition 3.3 *If $\varphi : X \multimap Y$ and $\psi : Y \multimap T$ are TSA_{C_D} maps then the composition $\psi \circ \varphi : X \multimap T$ is a TSA_{C_D} map.*

Proof. We have the following diagram:

$$X \xleftarrow{p_1} \Gamma_1 \xrightarrow{q_1} Y \xleftarrow{p_2} \Gamma_2 \xrightarrow{q_2} T,$$

where p_1, p_2 are cell-like maps and q_1, q_2 continuous maps. From assumption

$$\Gamma_1, \Gamma_2 \in C_D S C_L,$$

so the space

$$\Gamma = \{(\gamma_1, \gamma_2) \in \Gamma_1 \times \Gamma_2 : q_1(\gamma_1) = p_2(\gamma_2)\} \in C_D S C_L.$$

Hence and from Proposition 3.1 and Theorem 40.5 (see (2), p. 201) the proof is complete. □

In the field of the extension of mappings of TSA_{C_D}-type, there exists the following fact:

Theorem 3.4 (see (3; 7)) *Let $X \in C_D S C_L$, $A \subset X$ be closed in X ($A \neq X$) and let $Y \in ANR$ ($Y \in AR$). Assume that a multi-valued map $\varphi : A \multimap Y$ is TSA_{C_D}-type. Then φ has an elementary extension (see Definition 2.9) $\widetilde{\varphi} : U \multimap Y$ ($\widetilde{\varphi} : X \multimap Y$), where $U \supset A$ is some open set in X.*

Proof. The proof can be found in (3). □

From the previous Theorem it results that a multi-valued mapping of TSA_{C_D}-type can be extended in the same way that the single-valued ones can be extended and their extensions are single-valued as well. In the spaces that are separable, countably dimensional and locally compact, multi-valued mappings of TSA_{C_D}-type can also be used for the characterization of sets of ANR (AR) type. It is known that a space $X \in ANR$ ($X \in AR$) if there exist continuous mappings $f : U \to X$ ($f : E \to X$) and $g : X \to U$ ($g : X \to E$) such that for every $x \in X$ $f(g(x)) = x$, where $U \subset E$ is an open subset of some normed space E. If the mapping g is substituted with a multi-valued mapping of TSA_{C_D}-type, then the class of spaces of ANR (AR) type will not change.

Proposition 3.5 *Let $X \in C_DSC_L$, $X \in ANR$ ($X \in AR$) and let $A \subset X$ be a non-empty and closed set. If there exists a continuous map $f : U \to A$ ($f : E \to A$) and a u.s.c multi-valued map $\varphi : A \multimap U$ ($\varphi : A \multimap E$), where $U \subset E$ is some open set in a normed space E such that the following conditions are satisfied:*
$3.5.1$ *for each $x \in A$ $f(\varphi(x)) = \{x\}$,*
$3.5.2$ *φ has an elementary extension $\widetilde{\varphi} : V \multimap U$ ($\widetilde{\varphi} : X \multimap E$), where $V \supset A$ is some open set in X. Then $A \in ANR$ ($A \in AR$).*

Proof. We define a map $r : V \to A$ ($r : X \to A$) given by the formula:

$$r(x) = f(\widetilde{\varphi}(x)) \text{ for each } x \in V \ (x \in X),$$

where $\widetilde{\varphi} : V \multimap U$ ($\widetilde{\varphi} : X \multimap E$) is an elementary extension of φ and $V \supset A$ is some open set in X. From $3.5.2$ and $3.5.1$ the map r is a continuous function and for each $x \in A$ $r(x) = x$. Hence $A \in ANR$ ($A \in AR$). ☐

From the previous fact as well as from Theorem 3.4 and Proposition 2.3, the result is the following:
Proposition 3.6 *Let $X \in C_DSC_L$, $X \in ANR$ ($X \in AR$). Assume that a metrizable space Y is embedded as a closed set of a space X. If there exists a continuous map $f : U \to Y$ ($f : E \to Y$) and a u.s.c multi-valued map $\varphi : Y \multimap U$ ($\varphi : Y \multimap E$), where $U \subset E$ is some open set in a normed space E such that the following conditions are satisfied:*
$3.6.1$ *for each $y \in Y$ $f(\varphi(y)) = \{y\}$,*
$3.6.2$ *$\varphi \in$TSA$_{C_D}$.*
Then $Y \in ANR$ ($Y \in AR$).
In the field of homotopy (Definition 2.5), the Theorem on the extension of homotopy can be proven (see (4)). This results in the following theorems:

Theorem 3.7 *Let $Y \in AR$ and $Y \in C_DSC_L$. Assume that a metrizable space $X \in ANR$ is embedded as a closed set of a space Y. If X is TSA-contractible then $X \in AR$.*

Proof. Let $H : X \times [0, 1] \multimap X$ be a TSA - homotopy such that for any $x \in X$

$$H(x, 0) = \{x\} \text{ and } H(x, 1) = \{x_0\}, \tag{3.1}$$

where $x_0 \in X$ is a stationary point. From definition the map H is TSA- type, so there exist a space $Z \subset E$ (E is a normed space) and maps $p : Z \to X \times [0, 1] \subset Y \times [0, 1]$, $q : Z \to X$ such that p is a cell-like map and q is continuous and for any $(x, t) \in X \times [0, 1]$

$$q(p^{-1}(x, t)) \subset H(x, t) \text{ and} \tag{3.2}$$

$$q(p^{-1}(x, 1)) = \{x_0\}. \tag{3.3}$$

We define the map $\psi : (X \times [0, 1]) \cup (Y \times \{1\}) \multimap X$ given by formula:

$$\psi(x, t) = \begin{cases} H(x, t) & \text{for } (x, t) \in X \times [0, 1], \\ \{x_0\} & \text{for } (x, 1) \in Y \times \{1\}. \end{cases} \tag{3.4}$$

We show that $\psi \in TSA$. The map $\varphi : X \times \{1\} \multimap Z \times \{1\} \subset E \times \{1\}$ given by

$$\varphi(x,1) = p^{-1}(x,1) \times \{1\} \text{ for all } (x,1) \in X \times \{1\}$$

has an elementary extension (see Theorem 3.4) $\widetilde{\varphi} : Y \times \{1\} \multimap E \times \{1\}$. Let

$$A = \bigcup_{(x,t)\in X\times[0,1]} p^{-1}(x,t) \times \{t\}, \;\; B = E \times \{1\}, \;\; (A \cup B) \subset E \times [0,1].$$

We observe that

$$A \cap B = \bigcup_{x \in X} p^{-1}(x,1) \times \{1\}. \tag{3.5}$$

Let $\chi : (X \times [0,1]) \cup (Y \times \{1\})(A \cup B)$ be a map given by formula:

$$\chi(x,t) = \begin{cases} p^{-1}(x,t) \times \{t\} & \text{for } (x,t) \in X \times [0,1], \\ \widetilde{\varphi}(x,1) & \text{for } (x,1) \in Y \times \{1\}. \end{cases}$$

It is clear that for each $(x,t) \in (X \times [0,1]) \cup (Y \times \{1\})$ $\chi(x,t)$ is of a trivial shape. Let $\widetilde{q} : (A \cup B) \to X$ be a map given by formula:

$$\widetilde{q}(z,t) = \begin{cases} q(z) & \text{for } (z,t) \in A, \\ x_0 & \text{for } (z,1) \in B. \end{cases}$$

From (3.3) and (3.5) the map $\widetilde{q}$ is continuous. We observe that from (3.2) and (3.3) for any $(x,t) \in (X \times [0,1]) \cup (E \times \{1\})$

$$\widetilde{q}(\chi(x,t)) \subset \psi(x,t).$$

From Theorem 3.4 there exists an elementary extension

$$\widetilde{\psi} : U \multimap X \text{ of } \psi, \tag{3.6}$$

where $U \subset Y \times [0,1]$ is an open set in $Y \times [0,1]$ and $((X \times [0,1]) \cup (Y \times \{1\})) \subset U$. There exists an open set $V \subset Y$ such that $X \times [0,1] \subset V \times [0,1] \subset U$ and a Urysohn map $\lambda : Y \to [0,1]$ such that

$$\lambda(x) = 1 \text{ for any } x \in Y \backslash V \text{ and } \lambda(x) = 0 \text{ for any } x \in X. \tag{3.7}$$

Let $f : Y \to U$ (see (8), (8.2), p. 94) be a map given by formula

$$f(x) = (x, \lambda(x)) \text{ for all } x \in Y.$$

We define a map $r : Y \to X$ given by formula:

$$r(x) = \widetilde{\psi}(f(x)) \text{ for all } x \in Y.$$

We show that the map r is single-valued. If $x \in Y \backslash V$ then from (3.7), (3.4) and (3.1)

$$r(x) = \widetilde{\psi}(f(x)) = \widetilde{\psi}(x,1) = H(x,1) = x_0.$$

If $x \in X$ then from (3.7), (3.4) and (3.1) we have

$$r(x) = \widetilde{\psi}(f(x)) = \widetilde{\psi}(x,0) = H(x,0) = x. \tag{3.8}$$

If $x \in V \backslash X$ then from (3.7)

$$f(x) = (x, \lambda(x)) \in (V \times [0,1]) \backslash (X \times [0,1])$$

and from (3.6) (see Definition 2.9) the set

$$\widetilde{\psi}(f(x)) = \widetilde{\psi}(x, \lambda(x)) \text{ is single-valued.}$$

From (3.8) the map r is a retraction and the proof is complete. □

It is a known fact that if a space $X \in ANR$ (not necessarily finitely or countably dimensional) is contractible to a point, then $X \in AR$. If in the class of countably dimensional, separable and locally compact spaces contractibility is substituted with TSA-contractibility, then the result will be the same even though the notion of TSA-contractibility is essentially more general (see (4)). It is clear that if a compact space is of trivial shape (not necessarily contractible to a point), then it is TSA-contractible (see (4)). Finally, it should be noted that particularly multi-valued mappings $\varphi : \mathbb{R}^n \multimap \mathbb{R}^m$ of TSA_{C_D}-type, where $\mathbb{R}^n$ denotes an n-dimensional Euclidean space, have similar properties to the properties of single-valued mappings. Thanks to that, these mappings can be used for examining some properties of sets in Euclidean spaces.

4 Conclusions

In the article, it is shown that there exists some class of admissible mappings, $\varphi : \mathbb{R}^n \to \mathbb{R}^m$ that is closed due to a composition and has similar properties to the properties of single-valued mappings, e.g. in the range of fixed points, elementary extensions and the characterization of absolute retracts and absolute neighbourhood retracts. The main result of the article is Theorem 3.7 in which it is proven that a closed and non-empty set $X \subset \mathbb{R}^n$, $X \in ANR$, which is multivaluedly contractible to a point (not necessarily contractible to a point) is an absolute retract.

Competing Interests

The author declares that no competing interests exist.

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