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Positive Solutions for Systems of Singular Higher-Order Multi-Point Boundary Value Problems

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Authors' contributions

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Abstract

We study the existence of positive solutions of a system of higher-order nonlinear differential equations subject to multi-point boundary conditions, where the nonlinearities do not possess any sublinear or superlinear growth conditions and may be singular. In the proof of the main results, we use the Guo-Krasnosel'skii fixed point theorem.

Keywords: Higher-order differential system, singular equations, multi-point boundary conditions, positive solutions.

1 Introduction

We consider the system of higher-order singular ordinary differential equations

$$(S): \quad \begin{cases} u^{(n)}(t) + f\big(t, v(t)\big) = 0, & t \in (0,T), \\ v^{(m)}(t) + g\big(t, u(t)\big) = 0, & t \in (0,T), \end{cases}$$

with the multi-point boundary conditions

$$(BC): \quad \begin{cases} u(0) = \sum_{i=1}^{p} a_i u(\xi_i), \ \ u'(0) = \cdots = u^{(n-2)}(0) = 0, \ \ u(T) = \sum_{i=1}^{q} b_i u(\eta_i), \\ v(0) = \sum_{i=1}^{r} c_i v(\zeta_i), \ \ v'(0) = \cdots = v^{(m-2)}(0) = 0, \ \ v(T) = \sum_{i=1}^{l} d_i v(\rho_i), \end{cases}$$

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where $n,\ m \in \mathbb{N},\ n,\ m \geq 2,\ p, q, r, l \in \mathbb{N}$. In the case $n = 2$ or $m = 2$ the above conditions are of the form $u(0) = \sum_{i=1}^{p} a_i u(\xi_i),\ u(T) = \sum_{i=1}^{q} b_i u(\eta_i)$, or $v(0) = \sum_{i=1}^{r} c_i v(\zeta_i),\ v(T) = \sum_{i=1}^{l} d_i v(\rho_i)$, respectively, that is without conditions on the derivatives of u and v in the point 0.

We present some weaker assumptions on f and g, which do not possess any sublinear or superlinear growth conditions and may be singular at $t = 0$ and/or $t = T$, such that positive solutions for problem $(S) - (BC)$ exist. By a positive solution of $(S) - (BC)$, we understand a pair of functions $(u, v) \in (C([0,T]; \mathbb{R}_+) \cap C^n((0,T))) \times (C([0,T]; \mathbb{R}_+) \cap C^m((0,T)))$ satisfying (S) and (BC) with $\sup_{t\in[0,T]} u(t) > 0$, $\sup_{t\in[0,T]} v(t) > 0$. This problem is a generalization of the problem studied in [1], where in (BC) we have $a_i = 0$ for all $i = 1, \ldots, p$ and $c_i = 0$ for all $i = 1, \ldots, r$ (denoted by $(\widetilde{BC})$).

The system (S) with $n = m = 2$ and the boundary conditions $u(0) = 0,\ u(T) = \sum_{i=1}^{m-2} b_i u(\xi_i)$, $v(0) = 0,\ v(T) = \sum_{i=1}^{n-2} c_i v(\eta_i)$ has been investigated in [2]. In [3], the authors studied the existence of positive solutions for system (S) with $n = m = 2$ and the boundary conditions $u(0) = 0,\ u(1) = \alpha u(\eta),\ v(0) = 0, v(1) = \alpha v(\eta)$ with $\eta \in (0,1), 0 < \alpha\eta < 1$ $(T = 1)$. In [4], we investigated the existence and multiplicity of positive solutions for system (S) where f and g are nonsingular functions and the boundary conditions $(\widetilde{BC})$. The particular case of (S) with $n = m = 2, T = 1$ and boundary conditions which contain only one intermediate point has been studied in [5]. We also mention the paper [6], where the authors used the fixed point index theory to prove the existence of positive solutions for the system (S) with $f(t, v(t))$ and $g(t, u(t))$ replaced by $c(t)\tilde{f}(u(t), v(t))$ and $d(t)\tilde{g}(u(t), v(t))$, respectively, (with $\tilde{f}$ and $\tilde{g}$ singular functions) and $(\widetilde{BC})$ where $\frac{1}{2} \leq \eta_1 < \eta_2 < \cdots < \eta_q < 1$, $\frac{1}{2} \leq \rho_1 < \rho_2 < \cdots < \rho_l < 1$ $(T = 1)$. Some multi-point boundary value problems for systems of ordinary differential equations which involve positive eigenvalues were studied in recent years by using the Guo-Krasnosel'skii fixed point theorem. Namely, in [7], the authors give sufficient conditions for λ, μ, f and g $(f, g$ nonsingular functions) such that the system

$$(S_1): \quad \begin{cases} u^{(n)}(t) + \lambda a(t) f(u(t), v(t)) = 0, & t \in (0,T), \\ v^{(m)}(t) + \mu b(t) g(u(t), v(t)) = 0, & t \in (0,T), \end{cases}$$

with the boundary conditions (BC) has positive solutions. The system (S_1) with the boundary conditions $(\widetilde{BC})$ has been studied in [8]. The system (S_1) with $n = m = 2$ and the multi-point boundary conditions $\alpha u(0) - \beta u'(0) = 0,\ u(T) = \sum_{i=1}^{p-2} a_i u(\xi_i),\ \gamma v(0) - \delta v'(0) = 0,\ v(T) = \sum_{i=1}^{q-2} b_i v(\eta_i)$, has been investigated in [9]. Some particular cases of the above problems have been studied in [10-14].

In the last decades, nonlocal boundary value problems (including multi-point boundary value problems) for ordinary differential or difference equations/systems have become a rapidly growing area of research. Several phenomena in engineering, physics and life sciences can be modeled in this way. These problems have been studied by many authors by using different methods, such as fixed point theorems in cones, the Leray-Schauder continuation theorem, nonlinear alternatives of Leray-Schauder and coincidence degree theory.

In Section 2, we present some auxiliary results which investigate two boundary value problems for higher-order equations. In Section 3, we prove two existence results for the positive solutions with respect to a cone for our problem $(S)-(BC)$, which are based on the Guo-Krasnosel'skii fixed point theorem (see [15]) which is presented below.

Theorem 1. *Let X be a Banach space and let $C \subset X$ be a cone in X. Assume Ω_1 and Ω_2 are bounded open subsets of X with $0 \in \Omega_1 \subset \overline{\Omega_1} \subset \Omega_2$ and let $\mathcal{A}: C \cap (\overline{\Omega_2} \setminus \Omega_1) \to C$ be a completely continuous operator such that, either*

i) $\|\mathcal{A}u\| \le \|u\|,\ u \in C \cap \partial\Omega_1$, and $\|\mathcal{A}u\| \ge \|u\|,\ u \in C \cap \partial\Omega_2$, or
ii) $\|\mathcal{A}u\| \ge \|u\|,\ u \in C \cap \partial\Omega_1$, and $\|\mathcal{A}u\| \le \|u\|,\ u \in C \cap \partial\Omega_2$.

Then $\mathcal{A}$ has a fixed point in $C \cap (\overline{\Omega_2} \setminus \Omega_1)$.

2 Auxiliary Results

In this section, we present some auxiliary results from [7] and [16] related to the following n-order differential equation with multi-point boundary conditions

$$u^{(n)}(t) + y(t) = 0, \quad t \in (0,T), \tag{2.1}$$

$$u(0) = \sum_{i=1}^{p} a_i u(\xi_i), \ \ u'(0) = \cdots = u^{(n-2)}(0) = 0, \quad u(T) = \sum_{i=1}^{q} b_i u(\eta_i), \tag{2.2}$$

where $n \in \mathbb{N}$, $n \ge 2$, $p, q \in \mathbb{N}$. If $n = 2$, the condition (2.2) has the form $u(0) = \sum_{i=1}^{p} a_i u(\xi_i)$, $u(T) = \sum_{i=1}^{q} b_i u(\eta_i)$.

Lemma 2.1 *([7]) If $\Delta_1 = \left(1 - \sum_{i=1}^{q} b_i\right) \sum_{i=1}^{p} a_i \xi_i^{n-1} + \left(1 - \sum_{i=1}^{p} a_i\right)\left(T^{n-1} - \sum_{i=1}^{q} b_i \eta_i^{n-1}\right) \ne 0$, $0 < \xi_1 < \cdots < \xi_p < T$, $0 < \eta_1 < \cdots < \eta_q < T$ and $y \in C([0,T])$, then the solution of $(2.1)-(2.2)$ is given by*

$$\begin{aligned} u(t) = &-\int_0^t \frac{(t-s)^{n-1}}{(n-1)!} y(s)ds + \frac{t^{n-1}}{\Delta_1}\left\{\left(1 - \sum_{i=1}^{q} b_i\right)\sum_{i=1}^{p} a_i \int_0^{\xi_i} \frac{(\xi_i - s)^{n-1}}{(n-1)!} y(s)ds \right.\\ &\left. + \left(1 - \sum_{i=1}^{p} a_i\right)\frac{1}{(n-1)!}\left[\int_0^T (T-s)^{n-1} y(s)ds - \sum_{i=1}^{q} b_i \int_0^{\eta_i} (\eta_i - s)^{n-1} y(s)ds\right]\right\} \\ &+ \frac{1}{\Delta_1}\left\{\left(\sum_{i=1}^{p} a_i \xi_i^{n-1}\right)\frac{1}{(n-1)!}\left[\int_0^T (T-s)^{n-1} y(s)ds - \sum_{i=1}^{q} b_i \int_0^{\eta_i} (\eta_i - s)^{n-1} y(s)ds\right] \right.\\ &\left. - \left(T^{n-1} - \sum_{i=1}^{q} b_i \eta_i^{n-1}\right)\sum_{i=1}^{p} a_i \int_0^{\xi_i} \frac{(\xi_i - s)^{n-1}}{(n-1)!} y(s)ds\right\}. \end{aligned} \tag{2.3}$$

Lemma 2.2 *([7]) Under the assumptions of Lemma 2.1, the Green's function for the boundary value problem $(2.1)-(2.2)$ is given by*

$$\begin{aligned} G_1(t,s) = &\ g_1(t,s) \\ &+ \frac{1}{\Delta_1}\left[(T^{n-1} - t^{n-1})\left(1 - \sum_{i=1}^{q} b_i\right) + \sum_{i=1}^{q} b_i (T^{n-1} - \eta_i^{n-1})\right]\sum_{i=1}^{p} a_i g_1(\xi_i, s) \end{aligned}$$

$$+\frac{1}{\Delta_1}\left[t^{n-1}\left(1-\sum_{i=1}^{p} a_i\right)+\sum_{i=1}^{p} a_i\xi_i^{n-1}\right]\sum_{i=1}^{q} b_i g_1(\eta_i, s), \quad (t,s) \in [0,T]\times[0,T], \tag{2.4}$$

where

$$g_1(t,s) = \frac{1}{(n-1)!\,T^{n-1}}\begin{cases} t^{n-1}(T-s)^{n-1} - T^{n-1}(t-s)^{n-1}, & 0 \le s \le t \le T, \\ t^{n-1}(T-s)^{n-1}, & 0 \le t \le s \le T. \end{cases} \tag{2.5}$$

By using G_1, the solution u of problem $(2.1)-(2.2)$ given by (2.3) can be written as $u(t) = \int_0^T G_1(t,s)y(s)ds$.

Lemma 2.3 *([16]; see also [4]) The function* g_1 *given by* (2.5) *has the properties*

a) $g_1: [0,T]\times[0,T] \to \mathbb{R}_+$ *is a continuous function and* $g_1(t,s) \ge 0$ *for all* $(t,s) \in [0,T]\times[0,T]$.

b) $g_1(t,s) \le g_1(\theta_1(s), s)$, *for all* $(t,s) \in [0,T]\times[0,T]$.

c) For any $c \in (0, T/2)$, *we have* $\min_{t\in[c,T-c]} g_1(t,s) \ge \frac{c^{n-1}}{T^{n-1}}\, g_1(\theta_1(s), s)$, *for all* $s \in [0,T]$,

where $\theta_1(s) = s$ *if* $n = 2$ *and* $\theta_1(s) = \begin{cases} \frac{s}{1-\left(1-\frac{s}{T}\right)^{\frac{n-1}{n-2}}}, & s \in (0,T], \\ \frac{T(n-2)}{n-1}, & s = 0, \end{cases}$ *if* $n \ge 3$.

Lemma 2.4 *([7]) If* $a_i \ge 0$ *for all* $i = 1, \dots, p$, $\sum_{i=1}^{p} a_i < 1$, *and* $b_i \ge 0$ *for all* $i = 1, \dots, q$, $\sum_{i=1}^{q} b_i < 1$, $0 < \xi_1 < \cdots < \xi_p < T$, $0 < \eta_1 < \cdots < \eta_q < T$, *then the Green's function* G_1 *of the problem* $(2.1)-(2.2)$ *(given by* (2.4)*) is continuous on* $[0,T]\times[0,T]$ *and satisfies* $G_1(t,s) \ge 0$ *for all* $(t,s) \in [0,T]\times[0,T]$. *Moreover, if* $y \in C([0,T])$ *satisfies* $y(t) \ge 0$ *for all* $t \in [0,T]$, *then the unique solution* u *of problem* $(2.1)-(2.2)$ *satisfies* $u(t) \ge 0$ *for all* $t \in [0,T]$.

Lemma 2.5 *([7]) Assume that* $a_i \ge 0$ *for all* $i = 1, \dots, p$, $\sum_{i=1}^{p} a_i < 1$, *and* $b_i \ge 0$ *for all* $i = 1, \dots, q$, $\sum_{i=1}^{q} b_i < 1$, $0 < \xi_1 < \cdots < \xi_p < T$, $0 < \eta_1 < \cdots < \eta_q < T$. *Then the Green's function* G_1 *of the problem* $(2.1)-(2.2)$ *satisfies the inequalities*

a) $G_1(t,s) \le J_1(s)$, $\forall\, (t,s) \in [0,T]\times[0,T]$, *where*

$$J_1(s) = g_1(\theta_1(s), s) + \frac{1}{\Delta_1}\left[T^{n-1}\left(1-\sum_{j=1}^{q} b_j\right)+\sum_{i=1}^{q} b_i(T^{n-1}-\eta_i^{n-1})\right]\times$$
$$\times\sum_{i=1}^{p} a_i g_1(\xi_i, s) + \frac{1}{\Delta_1}\left[T^{n-1}\left(1-\sum_{j=1}^{p} a_j\right)+\sum_{i=1}^{p} a_i\xi_i^{n-1}\right]\sum_{i=1}^{q} b_i g_1(\eta_i, s). \tag{2.6}$$

b) For every $c \in (0, T/2)$, *we have*

$$\min_{t\in[c,T-c]} G_1(t,s) \ge \gamma_1 J_1(s) \ge \gamma_1\, G_1(t', s), \quad \forall\, t', s \in [0,T], \tag{2.7}$$

where $\gamma_1 = c^{n-1}/T^{n-1}$.

Lemma 2.6 *([7]) Assume that* $a_i \geq 0$ *for all* $i = 1, \dots, p$, $\sum_{i=1}^{p} a_i < 1$, *and* $b_i \geq 0$ *for all* $i = 1, \dots, q$, $\sum_{i=1}^{q} b_i < 1$, $0 < \xi_1 < \cdots < \xi_p < T$, $0 < \eta_1 < \cdots < \eta_q < T$, $c \in (0, T/2)$ *and* $y \in C([0,T])$, $y(t) \geq 0$ *for all* $t \in [0,T]$. *Then the solution* $u(t), t \in [0,T]$ *of problem* $(2.1) - (2.2)$ *satisfies the inequality* $\min_{t \in [c, T-c]} u(t) \geq \gamma_1 \max_{t' \in [0,T]} u(t')$.

We can also formulate similar results as Lemmas 2.1-2.6 above for the boundary value problem

$$v^{(m)}(t) + h(t) = 0, \quad t \in (0, T), \tag{2.8}$$

$$v(0) = \sum_{i=1}^{r} c_i v(\zeta_i), \;\; v'(0) = \cdots = v^{(m-2)}(0) = 0, \quad v(T) = \sum_{i=1}^{l} d_i v(\rho_i), \tag{2.9}$$

where $0 < \zeta_1 < \cdots < \zeta_r < T$, $c_i \geq 0$ for all $i = 1, \dots, r$, $0 < \rho_1 < \cdots < \rho_l < T$, $d_i \geq 0$ for all $i = 1, \dots, l$ and $h \in C([0,T])$. We denote by $\Delta_2, \gamma_2, g_2, \theta_2, G_2$ and J_2 the corresponding constants and functions for the problem $(2.8) - (2.9)$ defined in a similar manner as $\Delta_1, \gamma_1, g_1, \theta_1, G_1$ and J_1, respectively.

3 Main Results

In this section, we investigate the existence of positive solutions for our problem $(S) - (BC)$, under various assumptions on singular functions f and g.

We present the assumptions that we shall use in the sequel

(H_1) $0 < \xi_1 < \cdots < \xi_p < T$, $a_i \geq 0$ for all $i = 1, \dots, p$, $\sum_{i=1}^{p} a_i < 1$, $0 < \eta_1 < \cdots < \eta_q < T$, $b_i \geq 0$ for all $i = 1, \dots, q$, $\sum_{i=1}^{q} b_i < 1$, $0 < \zeta_1 < \cdots < \zeta_r < T$, $c_i \geq 0$ for all $i = 1, \dots, r$, $\sum_{i=1}^{r} c_i < 1$, $0 < \rho_1 < \cdots < \rho_l < T$, $d_i \geq 0$ for all $i = 1, \dots, l$, $\sum_{i=1}^{l} d_i < 1$.

(H_2) The functions $f, g \in C((0,T) \times \mathbb{R}_+, \mathbb{R}_+)$ and there exist $p_i \in C\big((0,T), \mathbb{R}_+\big)$, $q_i \in C(\mathbb{R}_+, \mathbb{R}_+)$, $i = 1, 2$, with $0 < \int_0^T p_i(t)dt < \infty$, $i = 1, 2$, $q_1(0) = 0$, $q_2(0) = 0$ such that

$$f(t,x) \leq p_1(t) q_1(x), \;\; g(t,x) \leq p_2(t) q_2(x), \;\; \forall\, t \in (0,T), \quad x \in \mathbb{R}_+.$$

(H_3) There exist $r_1, r_2 \in (0, \infty)$ with $r_1 r_2 \geq 1$ such that

$i)$ $q_{10}^{s} = \limsup_{x \to 0^+} \frac{q_1(x)}{x^{r_1}} \in [0, \infty)$; $\;\; ii)$ $q_{20}^{s} = \limsup_{x \to 0^+} \frac{q_2(x)}{x^{r_2}} = 0$.

(H_4) There exist $l_1, l_2 \in (0, \infty)$ with $l_1 l_2 \geq 1$ and $c \in (0, T/2)$ such that

$i)$ $f_\infty^{i} = \liminf_{x \to \infty} \inf_{t \in [c, T-c]} \frac{f(t,x)}{x^{l_1}} \in (0, \infty]$; $\;\; ii)$ $g_\infty^{i} = \liminf_{x \to \infty} \inf_{t \in [c, T-c]} \frac{g(t,x)}{x^{l_2}} = \infty$.

(H_5) There exist $\alpha_1, \alpha_2 \in (0, \infty)$ with $\alpha_1 \alpha_2 \leq 1$ such that

$i)$ $q_{1\infty}^{s} = \limsup_{x \to \infty} \frac{q_1(x)}{x^{\alpha_1}} \in [0, \infty)$; $\;\; ii)$ $q_{2\infty}^{s} = \limsup_{x \to \infty} \frac{q_2(x)}{x^{\alpha_2}} = 0$.

(H_6) There exist $\beta_1, \beta_2 \in (0,\infty)$ with $\beta_1\beta_2 \le 1$ and $c \in (0, T/2)$ such that

$i)$ $f_0^i = \liminf_{x\to 0^+} \inf_{t\in[c,T-c]} \frac{f(t,x)}{x^{\beta_1}} \in (0,\infty]$; $ii)$ $g_0^i = \liminf_{x\to 0^+} \inf_{t\in[c,T-c]} \frac{g(t,x)}{x^{\beta_2}} = \infty$.

The pair of functions $(u,v) \in (C([0,T]) \cap C^n((0,T))) \times (C([0,T]) \cap C^m((0,T)))$ is a solution for our problem $(S)-(BC)$ if and only if $(u,v) \in C([0,T]) \times C([0,T])$ is a solution for the nonlinear integral equations

$$\begin{cases} u(t) = \int_0^T G_1(t,s) f\big(s, v(s)\big) ds, \ \ t \in [0,T], \\ v(t) = \int_0^T G_2(t,s) g\big(s, u(s)\big) ds, \ \ t \in [0,T]. \end{cases} \tag{3.1}$$

The system (3.1) can be written as the nonlinear integral system

$$\begin{cases} u(t) = \int_0^T G_1(t,s) f\left(s, \int_0^T G_2(s,\tau) g(\tau, u(\tau))\, d\tau\right) ds, \ \ t \in [0,T], \\ \qquad\qquad v(t) = \int_0^T G_2(t,s) g\big(s, u(s)\big) ds, \ \ t \in [0,T]. \end{cases} \tag{3.2}$$

We consider the Banach space $X = C([0,T])$ with supremum norm $\|u\| = \sup_{t\in[0,T]} |u(t)|$ and define the cone $P \subset X$ by $P = \{u \in X,\ u(t) \ge 0,\ \forall\, t \in [0,T]\}$. For any $r > 0$, let $B_r = \{u \in C([0,T]),\ \|u\| < r\}$ and $\partial B_r = \{u \in C([0,T]),\ \|u\| = r\}$.

We also define the operator $\mathcal{A}: P \to X$ by

$$(\mathcal{A}u)(t) = \int_0^T G_1(t,s) f\left(s, \int_0^T G_2(s,\tau) g(\tau, u(\tau))\, d\tau\right) ds, \qquad t \in [0,T]. \tag{3.3}$$

Lemma 3.1 *Assume that $(H_1)-(H_2)$ hold. Then $\mathcal{A}: P \to P$ is completely continuous.*

Proof. We denote $\alpha = \int_0^T J_1(s) p_1(s)\, ds$ and $\beta = \int_0^T J_2(s) p_2(s)\, ds$. Using (H_2), we deduce that $0 < \alpha < \infty$ and $0 < \beta < \infty$. By Lemma 2.4 and the corresponding lemma for G_2, we get that $\mathcal{A}$ maps P into P.

We shall prove that $\mathcal{A}$ maps bounded sets into relatively compact sets. Suppose $D \subset P$ is an arbitrary bounded set. First we prove that $\mathcal{A}(D)$ is a bounded set. Because D is bounded, then there exists $M_1 > 0$ such that $\|u\| < M_1$ for all $u \in D$. By the continuity of q_2, there exists $M_2 > 0$ such that $M_2 = \sup_{x\in[0,M_1]} q_2(x)$. By using Lemma 2.5 for G_2, for any $u \in D$ and $s \in [0,T]$, we obtain

$$\int_0^T G_2(s,\tau) g(\tau, u(\tau))\, d\tau \le \int_0^T G_2(s,\tau) p_2(\tau)\, q_2(u(\tau))\, d\tau \le \beta M_2. \tag{3.4}$$

Because q_1 is continuous, there exists $M_3 > 0$ such that $M_3 = \sup_{x\in[0,\beta M_2]} q_1(x)$. Therefore, from (3.4), (H_2) and Lemma 2.5, we deduce

$$(\mathcal{A}u)(t) \le \int_0^T G_1(t,s)p_1(s)q_1\left(\int_0^T G_2(s,\tau)g\big(\tau,u(\tau)\big)d\tau\right) ds$$

$$\le M_3\int_0^T J_1(s)p_1(s)ds = \alpha M_3,\ \ \forall\ t\in[0,T]. \tag{3.5}$$

So, $\|\mathcal{A}u\| \le \alpha M_3$ for all $u \in D$. Therefore $\mathcal{A}(D)$ is bounded.

In what follows, we shall prove that $\mathcal{A}(D)$ is equicontinuous. By using Lemma 2.2, for all $t \in [0,T]$, we have

$$\begin{aligned}(\mathcal{A}u)(t) = \int_0^T \Big\{ g_1(t,s) + \frac{1}{\Delta_1}\Big[(T^{n-1}-t^{n-1})\Big(1-\sum_{i=1}^q b_i\Big) + \sum_{i=1}^q b_i(T^{n-1}-\eta_i^{n-1})\Big] \times \\ \times \sum_{i=1}^p a_i g_1(\xi_i,s) + \frac{1}{\Delta_1}\Big[t^{n-1}\Big(1-\sum_{i=1}^p a_i\Big) + \sum_{i=1}^p a_i\xi_i^{n-1}\Big]\sum_{i=1}^q b_i g_1(\eta_i,s)\Big\} \times \\ \times f\left(s,\int_0^T G_2(s,\tau)g(\tau,u(\tau))\,d\tau\right) ds = \frac{1}{(n-1)!\,T^{n-1}}\int_0^T [t^{n-1}(T-s)^{n-1} \\ -T^{n-1}(t-s)^{n-1}]f\left(s,\int_0^T G_2(s,\tau)g(\tau,u(\tau))\,d\tau\right) ds + \frac{1}{(n-1)!\,T^{n-1}}\int_t^T t^{n-1}(T-s)^{n-1} \times \\ \times f\left(s,\int_0^T G_2(s,\tau)g(\tau,u(\tau))\,d\tau\right) ds + \frac{1}{\Delta_1}\Big[(T^{n-1}-t^{n-1})\Big(1-\sum_{i=1}^q b_i\Big) \\ + \sum_{i=1}^q b_i(T^{n-1}-\eta_i^{n-1})\Big]\sum_{i=1}^p a_i\int_0^T g_1(\xi_i,s)f\left(s,\int_0^T G_2(s,\tau)g(\tau,u(\tau))\,d\tau\right) ds \\ + \frac{1}{\Delta_1}\Big[t^{n-1}\Big(1-\sum_{i=1}^p a_i\Big) + \sum_{i=1}^p a_i\xi_i^{n-1}\Big] \times \\ \times \sum_{i=1}^q b_i\int_0^T g_1(\eta_i,s)f\left(s,\int_0^T G_2(s,\tau)g(\tau,u(\tau))\,d\tau\right) ds.\end{aligned} \tag{3.6}$$

Therefore, we obtain

$$\begin{aligned}(\mathcal{A}u)'(t) = \int_0^t \frac{t^{n-2}(T-s)^{n-1}-T^{n-1}(t-s)^{n-2}}{(n-2)!\,T^{n-1}} f\left(s,\int_0^T G_2(s,\tau)g(\tau,u(\tau))\,d\tau\right) ds \\ + \int_t^T \frac{t^{n-2}(T-s)^{n-1}}{(n-2)!\,T^{n-1}} f\left(s,\int_0^T G_2(s,\tau)g(\tau,u(\tau))\,d\tau\right) ds + \frac{1}{\Delta_1}\Big[-(n-1)t^{n-2}\Big(1-\sum_{i=1}^q b_i\Big)\Big] \times \\ \times \sum_{i=1}^p a_i\int_0^T g_1(\xi_i,s)f\left(s,\int_0^T G_2(s,\tau)g(\tau,u(\tau))\,d\tau\right) ds + \frac{1}{\Delta_1}\Big[(n-1)t^{n-2}\Big(1-\sum_{i=1}^p a_i\Big)\Big] \times \\ \times \sum_{i=1}^q b_i\int_0^T g_1(\eta_i,s)f\left(s,\int_0^T G_2(s,\tau)g(\tau,u(\tau))\,d\tau\right) ds,\ \ \forall\ t\in(0,T).\end{aligned} \tag{3.7}$$

So, for any $t \in (0,T)$, we deduce

$$
\begin{aligned}
|(\mathcal{A}u)'(t)| \leq & \int_0^t \frac{t^{n-2}(T-s)^{n-1}+T^{n-1}(t-s)^{n-2}}{(n-2)!\,T^{n-1}} p_1(s) q_1\left(\int_0^T G_2(s,\tau) g(\tau,u(\tau))\,d\tau\right) ds \\
& + \int_t^T \frac{t^{n-2}(T-s)^{n-1}}{(n-2)!\,T^{n-1}} p_1(s) q_1\left(\int_0^T G_2(s,\tau) g(\tau,u(\tau))\,d\tau\right) ds \\
& + \frac{(n-1)t^{n-2}}{\Delta_1}\left(1-\sum_{i=1}^q b_i\right)\sum_{i=1}^p a_i \int_0^T g_1(\xi_i,s) p_1(s) q_1\left(\int_0^T G_2(s,\tau) g(\tau,u(\tau))\,d\tau\right) ds \\
& + \frac{(n-1)t^{n-2}}{\Delta_1}\left(1-\sum_{i=1}^p a_i\right)\sum_{i=1}^q b_i \int_0^T g_1(\eta_i,s) p_1(s) q_1\left(\int_0^T G_2(s,\tau) g(\tau,u(\tau))\,d\tau\right) ds \\
\leq & M_3\left(\int_0^t \frac{t^{n-2}(T-s)^{n-1}+T^{n-1}(t-s)^{n-2}}{(n-2)!\,T^{n-1}} p_1(s)\,ds + \int_t^T \frac{t^{n-2}(T-s)^{n-1}}{(n-2)!\,T^{n-1}} p_1(s)\,ds\right. \\
& + \frac{(n-1)t^{n-2}}{\Delta_1}\left(1-\sum_{i=1}^q b_i\right)\sum_{i=1}^p a_i \int_0^T g_1(\xi_i,s) p_1(s)\,ds + \frac{(n-1)t^{n-2}}{\Delta_1}\left(1-\sum_{i=1}^p a_i\right)\times \\
& \left.\times \sum_{i=1}^q b_i \int_0^T g_1(\eta_i,s) p_1(s)\,ds\right). \qquad (3.8)
\end{aligned}
$$

We denote

$$
h(t) = \int_0^t \frac{t^{n-2}(T-s)^{n-1}+T^{n-1}(t-s)^{n-2}}{(n-2)!\,T^{n-1}} p_1(s)\,ds + \int_t^T \frac{t^{n-2}(T-s)^{n-1}}{(n-2)!\,T^{n-1}} p_1(s)\,ds, \qquad (3.9)
$$

$$
\begin{aligned}
\mu(t) = & h(t) + \frac{(n-1)t^{n-2}}{\Delta_1}\left(1-\sum_{i=1}^q b_i\right)\sum_{i=1}^p a_i \int_0^T g_1(\xi_i,s) p_1(s)\,ds \\
& + \frac{(n-1)t^{n-2}}{\Delta_1}\left(1-\sum_{i=1}^p a_i\right)\sum_{i=1}^q b_i \int_0^T g_1(\eta_i,s) p_1(s)\,ds, \qquad t \in (0,T). \qquad (3.10)
\end{aligned}
$$

For the integral of the function h, by exchanging the order of integration, we obtain after some computations

$$
\begin{aligned}
\int_0^T h(t)dt = & \int_0^T \frac{(T-s)^{n-1}}{(n-2)!\,T^{n-1}}\left(\frac{T^{n-1}-s^{n-1}}{n-1}\right) p_1(s)\,ds + \int_0^T \frac{p_1(s)(T-s)^{n-1}}{(n-1)!}\,ds \\
& + \int_0^T \frac{(T-s)^{n-1}s^{n-1}}{(n-1)!\,T^{n-1}} p_1(s)\,ds = \frac{2}{(n-1)!}\int_0^T (T-s)^{n-1} p_1(s)\,ds < \infty. \qquad (3.11)
\end{aligned}
$$

For the integral of the function μ, we have

$$
\begin{aligned}
\int_0^T \mu(t)\,dt \leq & \frac{2}{(n-1)!}\int_0^T (T-s)^{n-1} p_1(s)\,ds + \frac{T^{n-1}}{\Delta_1}\left(1-\sum_{i=1}^q b_i\right)\times \\
& \times \sum_{i=1}^p a_i \int_0^T g_1(\theta_1(s),s) p_1(s)\,ds + \frac{T^{n-1}}{\Delta_1}\left(1-\sum_{i=1}^p a_i\right)\sum_{i=1}^q b_i \int_0^T g_1(\theta_1(s),s) p_1(s)\,ds \\
\leq & \frac{1}{(n-1)!}\left(2 + \frac{T^{n-1}}{\Delta_1}\left(1-\sum_{i=1}^q b_i\right)\left(\sum_{i=1}^p a_i\right) + \frac{T^{n-1}}{\Delta_1}\left(1-\sum_{i=1}^p a_i\right)\left(\sum_{i=1}^q b_i\right)\right)\times
\end{aligned}
$$

$$\times \int_0^T (T-s)^{n-1} p_1(s)\, ds < \infty. \tag{3.12}$$

We deduce that $\mu \in L^1(0,T)$. Thus for any given $t_1, t_2 \in [0,1]$ with $t_1 \le t_2$ and $u \in D$, by (3.8), we obtain

$$|(\mathcal{A}u)(t_1) - (\mathcal{A}u)(t_2)| = \left| \int_{t_1}^{t_2} (\mathcal{A}u)'(t) dt \right| \le M_3 \int_{t_1}^{t_2} \mu(t)\, dt. \tag{3.13}$$

From (3.12), (3.13) and absolute continuity of the integral function, we obtain that $\mathcal{A}(D)$ is equicontinuous. This conclusion together with (3.5) and Ascoli-Arzela theorem yields that $\mathcal{A}(D)$ is relatively compact. Therefore $\mathcal{A}$ is a compact operator.

By using similar arguments as those used in the proof of Lemma 2.4 from [3], we can show that $\mathcal{A}$ is continuous on P. Therefore $\mathcal{A}: P \to P$ is completely continuous. ∎

For $c \in (0, T/2)$, we define the cone

$$P_0 = \left\{ u \in X,\ u(t) \ge 0,\ \forall\, t \in [0,T], \min_{t \in [c, T-c]} u(t) \ge \gamma \|u\| \right\}, \tag{3.14}$$

where $\gamma = \min\{\gamma_1, \gamma_2\}$, γ_1 and γ_2 are given in Section 2. Under the assumptions (H_1), (H_2), we have $\mathcal{A}(P) \subset P_0$. Indeed, for $u \in P$, let $v = \mathcal{A}(u)$. By Lemma 2.6, we have $\min_{t \in [c, T-c]} v(t) \ge \gamma_1 \|v\| \ge \gamma \|v\|$, that is $v \in P_0$.

Theorem 3.1 *Assume that* $(H_1) - (H_4)$ *hold. Then the problem* $(S) - (BC)$ *has at least one positive solution* $\big(u(t), v(t)\big)$, $t \in [0,T]$.

Proof. We consider the cone P_0 with c given in (H_4). From $(H_3)\ i)$ and (H_2), we deduce that there exists $C_1 > 0$ such that

$$q_1(x) \le C_1 x^{r_1}, \quad \forall\, x \in [0,1]. \tag{3.15}$$

From $(H_3)\ ii)$ and (H_2), for $C_2 = \min\left\{ \big(1/(C_1 \alpha \beta^{r_1})\big)^{1/r_1}, 1/\beta \right\} > 0$ with α, β defined in the proof of Lemma 3.1, we conclude that there exists $\delta_1 \in (0,1)$ such that

$$q_2(x) \le C_2 x^{r_2}, \quad \forall\, x \in [0, \delta_1]. \tag{3.16}$$

From (3.16), (H_2) and Lemma 2.5, for any $u \in \partial B_{\delta_1} \cap P_0$ and $s \in [0,T]$, we obtain

$$\int_0^T G_2(s,\tau) g(\tau, u(\tau))\, d\tau \le C_2 \int_0^T J_2(\tau) p_2(\tau)\, d\tau \cdot \|u\|^{r_2} = C_2 \beta {\delta_1}^{r_2} \le {\delta_1}^{r_2} < 1. \tag{3.17}$$

By using $(3.15) - (3.17)$ and (H_2), for any $u \in \partial B_{\delta_1} \cap P_0$ and $t \in [0,T]$, we get

$$(\mathcal{A}u)(t) \le C_1 \int_0^T G_1(t,s)p_1(s)\left(\int_0^T G_2(s,\tau)g\big(\tau,u(\tau)\big)d\tau\right)^{r_1} ds$$
$$\le C_1 \int_0^T G_1(t,s)p_1(s)\left(C_2\int_0^T G_2(s,\tau)p_2(\tau)(u(\tau))^{r_2}\, d\tau\right)^{r_1} ds$$
$$\le C_1 \int_0^T J_1(s)p_1(s)ds \cdot \left(C_2\int_0^T J_2(\tau)p_2(\tau)d\tau\right)^{r_1} \cdot \|u\|^{r_1 r_2} \le \|u\|. \tag{3.18}$$

Therefore

$$\|\mathcal{A}u\| \le \|u\|, \qquad \forall\, u \in \partial B_{\delta_1} \cap P_0. \tag{3.19}$$

From $(H_4)\ i)$, we deduce that there exist $C_3 > 0$ and $x_1 > 0$ such that

$$f(t,x) \ge C_3 x^{l_1}, \qquad \forall\, x \ge x_1, \qquad \forall\, t \in [c, T-c]. \tag{3.20}$$

We consider now $C_4 = \max\left\{(\gamma_2\gamma^{l_2}\theta_2)^{-1}, \left(C_3\gamma_1\gamma_2^{l_1}\gamma^{l_1 l_2}\theta_1\theta_2^{l_1}\right)^{-1/l_1}\right\} > 0$, where $\theta_1 = \int_c^{T-c} J_1(s)ds > 0$ and $\theta_2 = \int_c^{T-c} J_2(s)ds > 0$. From $(H_4)\ ii)$, we conclude that there exists $x_2 \ge 1$ such that

$$g(t,x) \ge C_4 x^{l_2}, \qquad \forall\, x \ge x_2, \qquad \forall\, t \in [c, T-c]. \tag{3.21}$$

Now we choose $R_0 = \max\{x_1, x_2\}$ and $R > \max\left\{R_0/\gamma\,, R_0^{1/l_2}\right\}$. Then for any $u \in \partial B_R \cap P_0$, we have $\min_{t\in[c,T-c]} u(t) \ge \gamma\|u\| = \gamma R > R_0$.

By using (3.20) and (3.21), for any $u \in \partial B_R \cap P_0$ and $s \in [c, T-c]$, we obtain

$$\int_0^T G_2(s,\tau)g(\,\tau,u(\tau))\, d\tau \ge \gamma_2 C_4 \int_c^{T-c} J_2(\tau)\big(u(\tau)\big)^{l_2} d\tau$$

$$\ge \gamma_2 C_4 \gamma^{l_2} \int_c^{T-c} J_2(\tau)\, d\tau \cdot \|u\|^{l_2} \ge \|u\|^{l_2} = R^{l_2} > R_0. \tag{3.22}$$

Then for any $u \in \partial B_R \cap P_0$ and $t \in [c, T-c]$, we have

$$(\mathcal{A}u)(t) \ge \int_c^{T-c} G_1(t,s) f\left(s, \int_0^T G_2(s,\tau)g\big(\tau,u(\tau)\big)d\tau\right) ds$$
$$\ge C_3 \int_c^{T-c} G_1(t,s)\left(\gamma_2 \int_c^{T-c} J_2(\tau)C_4\big(u(\tau)\big)^{l_2} d\tau\right)^{l_1} ds$$
$$\ge C_3 C_4^{l_1}\gamma_2^{l_1} \int_c^{T-c} G_1(t,s)\gamma^{l_1 l_2}\|u\|^{l_1 l_2}\left(\int_c^{T-c} J_2(\tau)d\tau\right)^{l_1} ds$$
$$\ge C_3 C_4^{l_1}\gamma_2^{l_1}\gamma_1\gamma^{l_1 l_2}\left(\int_c^{T-c} J_1(s)ds\right)\left(\int_c^{T-c} J_2(\tau)d\tau\right)^{l_1} \|u\|^{l_1 l_2} \ge \|u\|. \tag{3.23}$$

Therefore we deduce

$$\|\mathcal{A}u\| \geq \|u\|, \qquad \forall\, u \in \partial B_R \cap P_0. \tag{3.24}$$

By (3.19), (3.24) and Theorem 1.1 i), we obtain that $\mathcal{A}$ has a fixed point $u_1 \in (\bar{B}_R \setminus B_{\delta_1}) \cap P_0$, that is $\delta_1 \leq \|u_1\| \leq R$. Let $v_1(t) = \int_0^T G_2(t,s) g(s, u_1(s)) ds$. Then $(u_1, v_1) \in P_0 \times P_0$ is a solution of $(S) - (BC)$. In addition $\|v_1\| > 0$. Indeed, if we suppose that $v_1(t) = 0$ for all $t \in [0,T]$, then by using (H_2) we have $f(s, v_1(s)) = f(s, 0) = 0$ for all $s \in [0,T]$. This implies $u_1(t) = 0$ for all $t \in [0,T]$, which contradicts $\|u_1\| > 0$. By using Theorem 1.1 from [17] (see also [18]), we obtain $u_1(t) > 0$ and $v_1(t) > 0$ for all $t \in (0, T-c]$. The proof of Theorem 3.1 is completed. ■

Theorem 3.2 *Assume that* (H_1), (H_2), (H_5) *and* (H_6) *hold. Then the problem* $(S) - (BC)$ *has at least one positive solution* $(u(t), v(t))$, $t \in [0,T]$.

Proof. We consider the cone P_0 with c given in (H_6). By $(H_5)\, i)$ we deduce that there exist $C_5 > 0$ and $C_6 > 0$ such that

$$q_1(x) \leq C_5 x^{\alpha_1} + C_6, \;\; \forall\, x \in [0,\infty). \tag{3.25}$$

From $(H_5)\, ii)$, for $\epsilon_0 > 0$, $\epsilon_0 < (2^{\alpha_1} C_5 \alpha \beta^{\alpha_1})^{-1/\alpha_1}$, we conclude that there exists $C_7 > 0$ such that

$$q_2(x) \leq \epsilon_0 x^{\alpha_2} + C_7, \;\; \forall\, x \in [0,\infty). \tag{3.26}$$

By using (3.25), (3.26) and (H_2), for any $u \in P_0$, we obtain

$$\begin{aligned}
(\mathcal{A}u)(t) &\leq \int_0^T G_1(t,s) p_1(s) q_1\left(\int_0^T G_2(s,\tau) g(\tau, u(\tau)) d\tau\right) ds \\
&\leq C_5 \int_0^T G_1(t,s) p_1(s) \left(\int_0^T G_2(s,\tau) g(\tau, u(\tau)) d\tau\right)^{\alpha_1} ds + C_6 \int_0^T J_1(s) p_1(s) ds \\
&\leq C_5 \int_0^T J_1(s) p_1(s) ds \left(\int_0^T J_2(\tau) p_2(\tau) d\tau\right)^{\alpha_1} (\epsilon_0 \|u\|^{\alpha_2} + C_7)^{\alpha_1} + \alpha C_6 \\
&\leq C_5 2^{\alpha_1} \epsilon_0^{\alpha_1} \alpha \beta^{\alpha_1} \|u\|^{\alpha_1 \alpha_2} + C_5 2^{\alpha_1} \alpha \beta^{\alpha_1} C_7^{\alpha_1} + \alpha C_6, \;\; \forall\, t \in [0,T].
\end{aligned} \tag{3.27}$$

By definition of ϵ_0, we can choose sufficiently large $R_1 > 0$ such that

$$\|\mathcal{A}u\| \leq \|u\|, \qquad \forall\, u \in \partial B_{R_1} \cap P_0. \tag{3.28}$$

From $(H_6)\, i)$, we deduce that there exist positive constants $C_8 > 0$ and $x_3 > 0$ such that $f(t,x) \geq C_8 x^{\beta_1}$, for all $x \in [0, x_3]$ and $t \in [c, T-c]$. From $(H_6)\, ii)$, for $\epsilon_1 = \left(C_8 \gamma_1 \gamma_2^{\beta_1} \gamma^{\beta_1 \beta_2} \theta_1 \theta_2^{\beta_1}\right)^{-1/\beta_1} > 0$, we conclude that there exists $x_4 > 0$ such that $g(t,x) \geq \epsilon_1 x^{\beta_2}$ for all $x \in [0, x_4]$ and $t \in [c, T-c]$.

We consider $x_5 = \min\{x_3, x_4\}$. So we obtain

$$f(t,x) \geq C_8 x^{\beta_1}, \qquad g(t,x) \geq \epsilon_1 x^{\beta_2}, \qquad \forall\, (t,x) \in [c, T-c] \times [0, x_5]. \tag{3.29}$$

From assumption $q_2(0) = 0$ and the continuity of q_2, we deduce that there exists sufficiently small $\epsilon_2 \in (0, \min\{x_5, 1\})$ such that $q_2(x) \leq \beta^{-1} x_5$ for all $x \in [0, \epsilon_2]$.

Therefore for any $u \in \partial B_{\epsilon_2} \cap P_0$ and $s \in [0, T]$, we have

$$\int_0^T G_2(s,\tau) g(\tau, u(\tau))\, d\tau \leq \beta^{-1} x_5 \int_0^T J_2(\tau) p_2(\tau) d\tau = x_5. \tag{3.30}$$

By (3.29), (3.30), Lemma 2.5 and Lemma 2.6, for any $t \in [c, T-c]$, we get

$$\begin{aligned}(\mathcal{A}u)(t) &\geq C_8 \int_c^{T-c} G_1(t,s) \left(\int_c^{T-c} G_2(s,\tau) g\big(\tau, u(\tau)\big) d\tau \right)^{\beta_1} ds \\ &\geq C_8 \gamma_1 \int_c^{T-c} J_1(s) \left[(\epsilon_1 \gamma_2)^{\beta_1} \left(\int_c^{T-c} J_2(\tau) \big(u(\tau)\big)^{\beta_2} d\tau \right)^{\beta_1} \right] ds \\ &\geq C_8 \gamma_1 \gamma_2^{\beta_1} \epsilon_1^{\beta_1} \gamma^{\beta_1 \beta_2} \theta_1 \theta_2^{\beta_1} \|u\|^{\beta_1 \beta_2} \geq \|u\|. \end{aligned} \tag{3.31}$$

Therefore

$$\|\mathcal{A}u\| \geq \|u\|, \qquad \forall\, u \in \partial B_{\epsilon_2} \cap P_0. \tag{3.32}$$

By (3.28), (3.32) and Theorem 1.1 ii), we deduce that $\mathcal{A}$ has at least one fixed point $u_2 \in \left(\bar{B}_{R_1} \setminus B_{\epsilon_2}\right) \cap P_0$. Then our problem $(S) - (BC)$ has at least one positive solution $(u_2, v_2) \in P_0 \times P_0$ where $v_2(t) = \int_0^T G_2(t,s) g\big(s, u_2(s)\big) ds$. The proof of Theorem 3.2 is completed.

4 Conclusions

We presented sufficient conditions on the functions (possibly singular) f and g such that the problem $(S) - (BC)$ has positive solutions.

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Competing Interests

Authors have declared that no competing interests exist.

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