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Nonlinear Perturbations of a Periodic Quasilinear Elliptic Problem with Discontinuous Nonlinearity in R^{N}[1]

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Abstract

Using variational methods, we estabilish existence of positive solutions for a class of quasilinear elliptic problems

$$-\Delta_p u + V(x)u^{p-1} = H(u-\beta)f(u), \text{ in } \mathbf{R}^N$$

where $\Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u), \beta > 0,\ 2 \le p < N,\ V$ is a positive, continuous perturbations of a periodic function, H is the Heaviside function and f is a continuous function with subcritical growth. The results of the semilinear equations are extended to the quasilinear problem.

Keywords: Nonlinear perturbations; discontinuous nonlinearity; quasilinear elliptic problem; periodic.

1 Introduction and basic results

In this paper, we study the existence of positive solution for the following class of quasilinear elliptic problem

$$\begin{cases} -\Delta_p u + V(x)u^{p-1} = H(u-\beta)f(u) & a.e. \text{ in } \mathbf{R}^N, \\ u > 0, & \text{in } \mathbf{R}^N, \end{cases} \qquad (P)_\beta$$

where $\Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u), 2 \le p < N,\ \beta \ge 0$ is a parameter, and H is the Heaviside function given by

$$H(t) = \begin{cases} 1 & \text{if } t > 0, \\ 0 & \text{if } t \le 0. \end{cases}$$

The function $V : \mathbf{R}^N \to \mathbf{R}$ is a positive continuous function satisfying:

$$\inf_{x \in \mathbf{R}^N} V(x) = V_0 > 0. \qquad (V_0)$$

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There is a $\mathbf{Z}^N$-periodic continuous function $V_m : \mathbf{R}^N \to \mathbf{R}$, that is,

$$V_m(x+y) = V_m(x) \quad \forall\, x \in \mathbf{R}^N,\ \ y \in \mathbf{Z}^N,$$

such that

$$V(x) \le V_m(x), \quad \forall\, x \in \mathbf{R}^N, \tag{V_1}$$

There is $x_1 \in \mathbf{R}^N$ such that

$$V(x_1) < V_m(x_1) \tag{V_2}$$

and

$$|V(x) - V_m(x)| \to 0, \ \text{ as } \ |x| \to +\infty. \tag{V_3}$$

The function $f : \mathbf{R} \to \mathbf{R}$ is a continuous function satisfying the following conditions
(f1) $\lim_{t\to 0^+} \frac{f(t)}{t^{p-1}} = 0.$
(f2) $\lim_{t\to +\infty} \frac{f(t)}{t^q} = 0$, for some $q \in (p-1, p^*-1)$, where $p^* = \frac{Np}{N-p}$.
(f3) There is $\theta \in (p, q+1)$ such that $0 < \theta F(t) \le f(t)t, \forall t > 0$, where $F(t) = \int_0^t f(s)ds.$
(f4) The function $t \to \frac{f(t)}{t^{p^*-1}}$ is increasing on $(0,\infty)$, which implies $t \to \frac{f(t)}{t^{p-1}}$ is also increasing on $(0,\infty)$.

The problem (1.1) appears in the study of non-Newtonian flows, chemotaxis, and biological pattern formation etc. For example, in the study of non-Newtonian flows, the constant p is a characteristic of medium. Media with $p > 2$ are called dilatant fluids and those with $p < 2$ are called pseudo-plastics. If $p = 2$ they are Newtonian fluids (see [32] and the references therein). The interest in the study of nonlinear partial differential equations with discontinuous nonlinearities has increased because many free boundary problems arising in mathematical physics may be stated in this form. Among these problems, we have the obstacle problem, the seepage surface problem, and the Elenbaas equation, see [2,14,16].

There are many papers having studied the problems with discontinuous nonlinearities, and we can refer the readers to [2,4,11,17-25] and the references therein. Several techniques have been developed or applied in these papers, such as variational methods for nondifferentiable functionals, sub and super solutions, and the theory of multivalued mappings.

When $p = 2$, the function V is periodic and f has a subcritical growth, the problem $(P)_\beta$ with $\beta = 0$ has been studied in [26], where the main tool used was the variational methods for C^1- functionals.

In [27], Alves , Marcos and Miyagaki discussed the existence of solution for the critical periodic and asymptotic periodic problem of the form

$$\begin{cases} -\triangle u + V(x)u = f(x,u) \ \text{ in } \ \mathbf{R}^2, \\ u \in H^1(\mathbf{R}^2), \quad u(x) > 0. \end{cases} \tag{1.1}$$

And in [8] they also established conditions for the existence of a positive solution for the periodic elliptic problem with critical growth, which given in the following form

$$\begin{cases} -\triangle u + V(x)u = \lambda u^q + u^p \ \text{ in } \ \mathbf{R}^N, \\ u \in H^1(\mathbf{R}^N), \quad u(x) > 0, \quad N \ge 3, \end{cases} \tag{1.2}$$

where $\lambda > 0$ is a parameter, $1 < q < p = 2^* - 1$, $2^* = \frac{2N}{N-2}$, and V is a positive continuous periodic function.

Recently, Alves and Nascimento [3] studied nonlinear perturbation of a periodic elliptic problem with discontinuous nonlinearity, which was given by

$$\begin{cases} -\triangle u + V(x)u = H(u-\beta)f(u) \quad a.e. \ \text{ in } \ \mathbf{R}^N, \\ u(x) > 0, \ \text{ in } \ \mathbf{R}^N. \end{cases} \tag{1.3}$$

For $p > 1$, Alves and Bertone [11] studied the following problem

$$-\Delta_p u = H(u-a)u^{p^*-1} + \lambda h(x) \quad x \in \mathbf{R}^N, \tag{1.4}$$

where H is the Heaviside function, $p^* = \frac{pN}{N-p}$ is the Sobolev critical exponent, and h is a positive function.

On the other hand, C.O.Alves et al.[28] also studied the following quasilinear problem

$$\begin{cases} -\Delta_p u + a(x)u^{p-1} = h(x)u^q + k(x)u^{p^*-1} \quad x \in \mathbf{R}^N, \\ u \in H^{1,p}(\mathbf{R}^N), \quad u(x) > 0, \quad x \in \mathbf{R}^N, \end{cases} \tag{1.5}$$

where $1 < p < N$, $p - 1 < q < p^* - 1$, $p^* = \frac{Np}{N-p}$ and a, h, k: $\mathbf{R}^N \to \mathbf{R}$ are continuous functions and there exist continuous Z-periodic functions A, H, K: $\mathbf{R}^N \to \mathbf{R}$ such that $a(x) \leq A(x)$, $h(x) \leq H(x)$, $k(x) \leq K(x)$, for all $x \in \mathbf{R}^N$, and

$$a(x) - A(x) \to 0, h(x) - H(x) \to 0, k(x) - K(x) \to 0, \text{as } \ |x| \to +\infty.$$

In the present paper, we will develop (1.3) into quasilinear one. Just as we know, when the nonlinear term is discontinuous and the function V is periodic, our first difficulty involving this class of problem is the fact that we can not use the classical variational methods, and it is necessary to use some results for Locally Lipschitz functional. Moreover, when the nonlinearity is continuous and satisfies some assumptions, the mountain pass level is equal to the minimum of the energy functional on Nehari Manifolds, which is a key point in a lot of papers. However, this property is not true for discontinuous nonlinearity. Hence, the arguments used in the above reference can not be repeated directly, and a careful analysis is necessary to get similar results to those found in [8,29,30].

By a modification of the method given in [3], we obtain the following main result.

Theorem 1.1. Assume that $(V_0) - (V_3)$ and $(f_1) - (f_4)$ hold. Then, there is $\beta^* > 0$, such that problem $(P)_\beta$ has a positive solution for all $\beta \in [0, \beta^*)$.

For the reader's convenience, we give the following some basic results according to O.Alves and Nascimento [3] and the references therein.

Theorem 1.2. [4] Let $I \in Lip_{loc}(X, R)$ with $I(0) = 0$ and satisfying:
(i) There are $r > 0$ and $\tau > 0$, such that $I(u) \geq \tau$, for $||u|| = r$, $u \in X$;
(ii) There is $e \in X \setminus B_r(0)$ with $I(e) < 0$.

If $c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t))$ and

$$\Gamma = \{\gamma \in C([0,1], X), \gamma(0) = 0, \text{ and } \ \gamma(1) = e\},$$

then $c \geq \tau$ and there is a sequence $\{u_n\} \subset X$ satisfying

$$I(u_n) \to c \quad \text{and} \quad \lambda(u_n) \to 0.$$

2 The periodic case

To prove Theorem 1.1, firstly, we need to establish the existence of solution for the periodic case. Thereby, in this section, we study the existence of positive solution for the following problem

$$\begin{cases} -\Delta_p u + V_m(x)u^{p-1} = H(u - \beta)f(u) \quad a.e. \ \text{ in } \ \mathbf{R}^N, \\ u > 0, \qquad\qquad \text{in } \ \mathbf{R}^N, \end{cases} \tag{$(P)_{m,\beta}$}$$

Hereafter, we consider the Sobolev space $W^{1,p}(\mathbf{R}^N)$ endowed with the norm

$$||u||^p = \int_{\mathbf{R}^N} (|\nabla u|^p + V_m(x)u^p).$$

A direct computation shows that this norm is equivalent to the usual norm in $W^{1,p}(\mathbf{R}^N)$.

The energy functional related to $(P)_{m,\beta}$ is given by

$$I_{m,\beta}(u) = \frac{1}{p}\int_{\mathbf{R}^N}(|\nabla u|^p + V_m(x)u^p) - \int_{\mathbf{R}^N} G(u),$$

where $G(t) = \int_0^t g(\tau)d\tau$ and $g(t) = H(t-\beta)f(t)$.

From now on, since we intend to find positive solutions for problem $(P)_{m,\beta}$, we will suppose that

$$f(t) = 0, \qquad \forall t \leq 0. \tag{2.1}$$

Using the same methods as [3], we have the following lemmas

Lemma 2.1. The functional $I_{p,\beta}$ verifies the mountain pass geometry, that is, the conditions (i) and (ii) of Theorem 1.2 hold.

From Lemma 2.1 and Theorem 1.2 there is a sequence $\{u_n\} \subset W^{1,p}(\mathbf{R}^N)$ satisfying

$$I_{p,\beta}(u_n) \to c_{p,\beta} \;\text{ and }\; \lambda_{p,\beta}(u_n) \to 0, \tag{2.2}$$

where $c_{p,\beta}$ is the mountain pass level for $I_{p,\beta}$.

Lemma 2.2. The sequence $\{u_n\}$ is bounded in $W^{1,p}(\mathbf{R}^N)$.

Lemma 2.3. Let $\{u_n\} \subset W^{1,p}(\mathbf{R}^N)$ be the sequence given in (2.2), then for each $R > 0$, there are $\{y_n\} \subset \mathbf{R}^N$ and $\alpha > 0$ satisfying

$$\int_{B_R(y_n)} |u_n|^p dx \geq \alpha \qquad \forall n \in N.$$

Remark 2.1. The sequence $\{y_n\}$ given in Lemma 2.3 can be chosen in Z^N, and we can find details in [8].

Using the same methods as Proposition 3.1 of [3], we have the following lemma

Lemma 2.4. Let $v_n(x) = u_n(x + y_n)$, where $\{y_n\}$ is given by Lemma 2.3. Then,

$$I_{m,\beta}(v_n) \to c_{m,\beta} \quad and \quad \lambda_{m,\beta}(v_n) \to 0.$$

Theorem 2.1. Assume that $(V_0) - (V_3)$ and $(f1) - (f4)$ hold. Then, there is $\beta^* > 0$ such that problem $(P)_{m,\beta}$ possesses a positive solution for all $\beta \in [0, \beta^*)$.

Proof. From Lemma 2.2, there is a subsequence of $\{u_n\}$, still denoted by itself and there exists $u \in W^{1,p}(\mathbf{R}^N)$ such that

$$u_n \rightharpoonup u \quad \text{in} \quad W^{1,p}(\mathbf{R}^N),$$

and

$$u_n \to u \quad \text{in} \quad L^s_{loc}(\mathbf{R}^N), \quad 1 \leq s < p^*,$$
$$u_n \to u \quad a.e \quad \text{in} \quad \mathbf{R}^N.$$

Thus, the sequence $\{v_n\}$ given by Lemma 2.4 is also bounded and there is $v \in W^{1,p}(\mathbf{R}^N)$ such that

$$v_n \rightharpoonup v \quad \text{in} \quad W^{1,p}(\mathbf{R}^N),$$

and

$$v_n \to v \quad \text{in} \quad L^s_{loc}(\mathbf{R}^N), \quad 1 \le s < p^*,$$
$$v_n \to v \quad a.e \quad \text{in} \quad \mathbf{R}^N.$$

Combining the Sobolev embedding with Lemma 2.3, we get

$$\int_{B_R(0)} v^p = \lim_{n\to\infty} \inf \int_{B_R(0)} v_n^p = \lim_{n\to\infty} \inf \int_{B_R(y_n)} u_n^p \ge \alpha > 0,$$

showing that $v \not\equiv 0$. On the other hand, from (2.1), we can assume without loss of generality that $v_n \ge 0$. Then, $v(x) \ge 0, \forall x \in \mathbf{R}^N$ and $v \not\equiv 0$.

Next, we will show that v is a nontrivial solution for $(P)_{m,\beta}$.

In fact, we have prove that $v \not\equiv 0$, thus, we need to show that v satisfies

$$-\Delta_p v + V_m(x)v^{p-1} = H(v-\beta)f(v) \quad a.e. \quad \text{in} \quad \mathbf{R}^N.$$

Once that $\{v_n\}$ satisfies

$$I_{m,\beta}(v_n) \to c_{m,\beta} \quad \text{and} \quad \lambda_{m,\beta}(v_n) \to 0,$$

we know that there are $\{\omega_n\} \subset \partial I_{m,\beta}(v_n)$ and $\{\rho_n\} \subset \partial\Psi_\beta(v_n)$ such that $||\omega_n||_* \to 0$ and

$$\langle \omega_n, \phi\rangle = \int_{\mathbf{R}^N} (|\nabla v_n|^{p-2}\nabla v_n \nabla\phi + V_m(x)v_n^{p-1}\phi) - \int_{\mathbf{R}^N} \rho_n \phi, \quad \forall \phi \in W^{1,p}(\mathbf{R}^N) \tag{2.3}$$

with

$$\rho_n(x) \in [\underline{g}(v_n(x)), \overline{g}(v_n(x))] \quad a.e. \quad \text{in} \quad \mathbf{R}^N. \tag{2.4}$$

From (2.4) and $\{\rho_n\}$ is bounded in $L^{\frac{q+1}{q}}(\mathbf{R}^N)$, there is $C_1 > 0$ such that

$$|\rho_n| \le C_1 |v_n|^q, \quad \forall n \in N.$$

from where it follows that

$$|\rho_n|^{\frac{q+1}{q}}_{L^{\frac{q+1}{q}}(\mathbf{R}^N)} \le C_2 \int_{\mathbf{R}^N} |v_n|^{q+1}.$$

Since $\{v_n\}$ is bounded in $L^{q+1}(\mathbf{R}^N)$, there is $M > 0$ such that

$$|\rho_n|_{L^{\frac{q+1}{q}}(\mathbf{R}^N)} \le M, \quad \forall n \in N.$$

Consequently, there is $\rho_0 \in L^{\frac{q+1}{q}}(\mathbf{R}^N)$ satisfying

$$\rho_n \rightharpoonup \rho_0 \quad \text{in} \quad L^{\frac{q+1}{q}}(\mathbf{R}^N), \tag{2.5}$$

or equivalently,

$$\int_{\mathbf{R}^N} \rho_n \phi \to \int_{\mathbf{R}^N} \rho_0 \phi, \quad \forall \phi \in L^{q+1}(\mathbf{R}^N). \tag{2.6}$$

Letting $n \to \infty$ in (2.3) and using (2.6), we obtain the identity

$$\int_{\mathbf{R}^N} (|\nabla v|^{p-2}\nabla v \nabla\phi + V_m(x)v^{p-1}\phi) = \int_{\mathbf{R}^N} \rho_0 \phi, \quad \forall \phi \in W^{1,p}(\mathbf{R}^N),$$

which yields v is a weak solution of the problem

$$\begin{cases} -\Delta_p v + V_m(x)v^{p-1} = \rho_0 & \text{in} \quad \mathbf{R}^N, \\ u > 0, & \text{in} \quad \mathbf{R}^N. \end{cases} \tag{$(P)_0$}$$

Repeating the similar arguments explored in [10,11], we can get that

$$\rho_n(x) \to \rho_0 \quad a.e. \quad \text{in} \quad \mathbf{R}^N. \tag{2.7}$$

Final, we will show that there exists $\beta^* > 0$ such that $\Gamma_\beta = \{x \in \mathbf{R}^N : v(x) = \beta\}$ has null measure for all $\beta \in (0, \beta^*)$.

In fact, we know that

$$\rho_n(x) \leq \overline{g}(v_n(x)) \quad a.e. \text{ in } \mathbf{R}^N.$$

Consequently, from definition of $\overline{g}$, we have

$$\rho_n(x) \leq f(v_n(x)) \quad a.e. \text{ in } \mathbf{R}^N.$$

Therefore, for all nonnegative function $\phi \in L^{q+1}(\mathbf{R}^N)$,

$$\int_{\mathbf{R}^N} \rho_n \phi \leq \int_{\mathbf{R}^N} f(v_n)\phi.$$

The weak limit

$$\rho_n \rightharpoonup \rho_0 \text{ in } L^{\frac{q+1}{q}}(\mathbf{R}^N)$$

together with the Sobolev embedding gives

$$\int_{\mathbf{R}^N} \rho_0 \phi \leq \int_{\mathbf{R}^N} f(v)\phi.$$

Since ϕ is arbitrary, we obtain

$$\rho_0(x) \leq f(v(x)) \quad a.e. \text{ in } \mathbf{R}^N.$$

From this, we have

$$\rho_0(x) \leq f(\beta) \quad a.e. \text{ in } \Gamma_\beta. \tag{2.8}$$

Now, if Γ_β has a positive measure, by using the Morrey-Stampacchia's Theorem (see [12,13]), we have that

$$-\bigtriangleup_p v(x) = 0 \quad a.e. \text{ in } \Gamma_\beta.$$

Thereby, we have

$$V_m(x)\beta^{p-1} \leq f(\beta) \quad a.e. \text{ in } \Gamma_\beta,$$

and thus

$$V_0 \leq \frac{f(\beta)}{\beta^{p-1}}.$$

By (f1), we have

$$\frac{f(\beta)}{\beta^{p-1}} \to 0, \quad \text{as } \beta \to 0.$$

Hence, we can conclude that there is $\beta^* > 0$ such that $\Gamma_\beta = \{x \in \mathbf{R}^N : v(x) = \beta\}$ has null measure for all $\beta \in (0, \beta^*)$.

Now, for $\beta \in (0, \beta^*)$ and from the following limit

$$v_n(x) \to v(x) \quad a.e. \text{ in } \mathbf{R}^N,$$

and (2.4), we can see that

$$\rho_n(x) \to H(v(x) - \beta)f(v(x)) \quad a.e. \text{ in } \mathbf{R}^N. \tag{2.9}$$

The last limit combined with (3.11) yields

$$\rho_0 = H(v(x) - \beta)f(v(x)) \quad a.e. \text{ in } \mathbf{R}^N.$$

Moreover, since the set Γ_β has null measure, we see that $v(x)$ satisfies $(P)_{m,\beta}$ in the "strong" sense, i.e.,

$$-\Delta_p v + V_m(x)v^{p-1} = H(v(x) - \beta)f(v(x)) \quad a.e. \text{ in } \mathbf{R}^N.$$

The positivity of v follows by maximum principles. Therefore, we have showed that v is a nontrivial solution for $(P)_{m,\beta}$.

3 The General Case

In this section, we will prove the existence of solution for problem $(P)_\beta$. Moreover, we denote by I_0, $I_{m,0} : W^{1,p}(\mathbf{R}^N) \to \mathbf{R}$ the energy functional associated with the problems $(P)_0$ and $(P)_{m,0}$, which are given by

$$I_0(u) = \frac{1}{p}\int_{\mathbf{R}^N} (|\nabla u|^p + V(x)u^p) - \int_{\mathbf{R}^N} F(u),$$

and

$$I_{m,0}(u) = \frac{1}{p}\int_{\mathbf{R}^N} (|\nabla u|^p + V_m(x)u^p) - \int_{\mathbf{R}^N} F(u).$$

These functionals are C^1 and it is well known that there are ψ_m, $\varphi \in W^{1,p}(\mathbf{R}^N)$ such that

$$I_{m,0}(\psi_m) = c_{m,0} \quad \text{and} \quad I'_{m,0}(\psi_m) = 0,$$

and

$$I_0(\varphi) = c_0, \quad \text{and} \quad I'_0(\varphi) = 0,$$

where $c_{m,0}$ and c_0 denote the mountain pass levels. In addition, from $(V_1) - (V_2)$, we can prove that $c_0 < c_{m,0}$. Since its proof follows by using the similar methods used in [8], we omit it here. Thereby, we can find $R > 0$ and $\delta > 0$ satisfying

$$I_0(\varphi_R) < c_0 + \frac{\delta}{4} \quad \text{and} \quad c_0 + 2\delta < c_{m,0} \tag{3.1}$$

where $\varphi_R = \eta(\frac{x}{R})\varphi$ and $\eta \in C_0^\infty(\mathbf{R}^N)$ satisfies

$$0 \le \eta \le 1, \quad \eta(x) = 1, \quad \forall x \in B_1(0) \quad \text{and} \quad \eta(x) = 0. \quad \forall x \in B_2^c(0).$$

According to Lebesgue's Theorem,

$$\varphi_R \to \varphi \quad \in W^{1,p}(\mathbf{R}^N) \quad \text{as} \quad R \to +\infty.$$

Consequently, $\varphi_R \not\equiv 0$ for R large enough. Moreover, there is $t_R > 0$ with $\lim_{R\to+\infty} t_R = 1$ such that

$$I_0(t_R\varphi_R) = \max_{t\ge 0} I_0(t\varphi_R).$$

Thereby, we choose $R > 0$ large enough satisfying

$$I_0(t_R\varphi_R) < c_0 + \frac{\delta}{4}.$$

Furthermore, for each $\beta > 0$, the energy functionals associated with $(P)_m$ and $(P)_{m,\beta}$ will be denoted by

$$I_\beta(u) = \frac{1}{p}\int_{\mathbf{R}^N} (|\nabla u|^p + V(x)u^p) - \int_{\mathbf{R}^N} G(u),$$

and

$$I_{m,\beta}(u) = \frac{1}{p}\int_{\mathbf{R}^N} (|\nabla u|^p + V_m(x)u^p) - \int_{\mathbf{R}^N} G(u),$$

respectively. In what follows, let us denote by c_β, $c_{m,\beta}$ the mountain pass levels of these functionals.

Using the same methods as Lemma 4.1 of [3], we have the following lemma

Lemma 3.1. There exists $\beta^* > 0$ such that $c_\beta < c_{m,\beta}$ for all $\beta \in (0, \beta^*)$.

Since I_β verifies the mountain pass geometry, there is a sequence $\{u_n\} \subset W^{1,p}(\mathbf{R}^N)$ satisfying

$$I_\beta(u_n) \to c_\beta \quad \text{and} \quad \lambda_\beta(u_n) \to 0.$$

Using the same arguments explored in Sect.3, we can get that $\{u_n\}$ is bounded in $W^{1,p}(\mathbf{R}^N)$, and there is $u \in W^{1,p}(\mathbf{R}^N)$ such that

$$u_n \rightharpoonup u \quad \text{in} \quad W^{1,p}(\mathbf{R}^N),$$

and

$$u_n \to u \quad \text{in} \quad L^s_{loc}(\mathbf{R}^N), \quad 1 \le s < p^*,$$
$$u_n \to u \quad a.e \quad \text{in} \quad \mathbf{R}^N.$$

Hereafter, we will use the following notations for the functionals I_β and $I_{m,\beta}$:

$$I_\beta(u) = Q(u) - \Psi_\beta(u)$$

and

$$I_{m,\beta}(u) = Q_m(u) - \Psi_\beta(u),$$

where

$$Q(u) = \frac{1}{p}\int_{\mathbf{R}^N}(|\nabla u|^p + V(x)u^p), \quad Q_m(u) = \frac{1}{p}\int_{\mathbf{R}^N}(|\nabla u|^p + V_m(x)u^p)$$

and

$$\Psi_\beta(u) = \int_{\mathbf{R}^N} G(u)$$

.

Lemma 3.2. The weak limit u is nontrivial, that is $u \not\equiv 0$.

Proof. Arguing by contradiction, we suppose that $u = 0$. Then,

$$u_n \rightharpoonup 0 \quad \text{in} \quad W^{1,p}(\mathbf{R}^N). \tag{3.2}$$

By (V_3), given $r > 0$ and $\epsilon > 0$, there is $n_0 \in \mathbf{R}^N$ such that

$$\int_{|x|\ge r} |V(x) - V_m(x)|u_n^p \le \epsilon, \qquad \forall n \ge n_0. \tag{3.3}$$

On the other hand, once that

$$u_n \to 0 \quad \text{in} \quad L^s_{loc}(\mathbf{R}^N), \quad 1 \le s < p^*,$$

it follows that

$$\lim_{n\to+\infty}\int_{|x|\le r} |V(x) - V_m(x)|u_n^s = 0 \tag{3.4}$$

and in particular,

$$\lim_{n\to+\infty}\int_{|x|\le r} |V(x) - V_m(x)|u_n^p = 0. \tag{3.5}$$

From (3.3) and (3.5), we have

$$\lim_{n\to+\infty}\int_{\mathbf{R}^N} |V(x) - V_m(x)|u_n^p = 0.$$

The above limit implies that

$$Q_m(u_n) = Q(u_n) + o_n(1)$$

and

$$Q'_m(u_n) = Q'(u_n) + o_n(1).$$

Hence, we have

$$I_{m,\beta}(u_n) = I_\beta(u_n) + o_n(1) \to c_\beta$$

and

$$\lambda_{m,\beta} \to 0.$$

From Lemma 3.2 and Remark 2.1, there are $\{z_n\} \subset Z^N$, $r_1 > 0$ and $\eta_1 > 0$ satisfying

$$\int_{B_{r_1}(z_n)} u_n^p \geq \eta_1 > 0.$$

Setting $v_n(x) = u_n(x + z_n)$ and repeating the same arguments explored in Sect.2, we have $\{v_n\}$ is bounded in $W^{1,p}(\mathbf{R}^N)$. Thus, there is $v \in W^{1,p}(\mathbf{R}^N \setminus \{0\})$ and a subsequence of $\{v_n\}$, still denoted by itself, such that

$$v_n(x) \rightharpoonup v \quad \text{in } W^{1,p}(\mathbf{R}^N)$$

with

$$I_{m,\beta}(v_n) \to c_{m,\beta} \quad \text{and} \quad \lambda_{m,\beta}(v_n) \to 0.$$

Moreover, v is a positive solution of the problem

$$-\Delta_p v + V_m(x)v^{p-1} = H(v(x) - \beta)f(v(x)) \quad a.e. \text{ in } \mathbf{R}^N, \tag{3.6}$$

with

$$\text{med}(\Gamma_\beta) = 0.$$

In what follows, let $t > 0$ satisfying

$$tv \in N_{m,0} = \{u \in W^{1,p}(\mathbf{R}^N) \setminus \{0\}; I'_{m,0}(u)u = 0\},$$

where $N_{m,0}$ is the Nehari manifold associated with $I_{m,0}$. Then,

$$\int_{\mathbf{R}^N} (|\nabla v|^p + V_m(x)v^p) = \int_{\mathbf{R}^N} \frac{f(tv)tv}{t^p} = \int_{\mathbf{R}^N} \frac{f(tv)v}{t^{p-1}} \tag{3.7}$$

Once that,

$$\int_{\mathbf{R}^N} (|\nabla v|^p + V_m(x)v^p) < \int_{\mathbf{R}^N} f(v)v, \tag{3.8}$$

the condition (f4) together with (3.7) and (3.8) yields $t \in (0, 1)$. Additionally, the below characterization

$$c_{m,0} = \inf_{u \in N_{m,0}} I_{m,0}(u) (\text{see}[9]),$$

gives the inequality

$$c_{m,0} \leq I_{m,0}(tv),$$

which leads to

$$c_{m,0} \leq t^p(\frac{1}{p} - \frac{1}{q+1})||v||^p + \int_{\mathbf{R}^N} [\frac{1}{q+1} f(tv)tv - F(tv)]$$

By (f4), the function

$$h(s) = \frac{1}{q+1} f(s)s - F(s)$$

is nondecreasing, then,

$$c_{m,0} < (\frac{1}{p} - \frac{1}{q+1})||v||^p + \int_{\mathbf{R}^N} [\frac{1}{q+1} f(v)v - F(v)],$$

or equivalently

$$c_{m,0} < (\frac{1}{p} - \frac{1}{q+1})||v||^p + \int_{[v \leq \beta]} [\frac{1}{q+1} f(v)v - F(v)] + \int_{[v > \beta]} [\frac{1}{q+1} f(v)v - F(v)].$$

From Lebesgue's Theorem, we have

$$\int_{[v\leq\beta]}[\frac{1}{q+1}f(v)v - F(v)] \to 0 \quad \text{as} \quad \beta \to 0.$$

Hence, there is $\beta^* > 0$ such that

$$\int_{[v\leq\beta]}[\frac{1}{q+1}f(v)v - F(v)] < \frac{\delta}{8}, \quad \forall\beta \in (0, \beta^*).$$

Thereby,

$$c_{m,0} < (\frac{1}{p} - \frac{1}{q+1})||v||^p + \int_{[v>\beta]}[\frac{1}{q+1}f(v)v - F(v)] + \frac{\delta}{8} \quad \forall\beta \in (0, \beta^*).$$

Combining the fact that v is a solution of (4.6) with med$(\Gamma_\beta) = 0$, we get

$$(\frac{1}{p} - \frac{1}{q+1})||v||^p + \int_{[v>\beta]}[\frac{1}{q+1}f(v)v - F(v)] \leq \int_{[v>\beta]}[\frac{1}{p}f(v)v - F(v)].$$

Consequently,

$$c_{m,0} < \int_{[v>\beta]}[\frac{1}{p}f(v)v - F(v)] + \frac{\delta}{8} \quad \forall\beta \in (0, \beta^*). \tag{3.9}$$

Recalling that $\langle \omega_n, v_n \rangle = o_n(1)$, we have

$$I_{m,\beta}(v_n) \quad = \quad I_{m,\beta}(v_n) - \frac{1}{p}\langle \omega_n, v_n \rangle + o_n(1) = \int_{\mathbf{R}^N}[\frac{1}{p}\rho_n v_n - G(v_n)] + o_n(1).$$

Since

$$\int_{[v_n<\beta]}[\frac{1}{p}\rho_n v_n - G(v_n)] = 0,$$

it follows that

$$I_{m,\beta}(v_n) = \int_{[v_n\geq\beta]}[\frac{1}{p}\rho_n v_n - G(v_n)] = \int_{[v_n\geq\beta]}[\frac{1}{p}f(v_n)v_n - G(v_n)].$$

Now, the below inequality

$$-G(t) = -F(t) + F(\beta) \geq -F(t), \quad \forall t \geq 0,$$

leads to

$$I_{m,\beta}(v_n) = \int_{[v_n\geq\beta]}[\frac{1}{p}f(v_n)v_n - G(v_n)] \geq \int_{[v_n\geq\beta]}[\frac{1}{p}f(v_n)v_n - F(v_n)].$$

The condition (f3) together with Fatou's Lemma and med$(\Gamma_\beta) = 0$ implies that

$$\lim_{n\to\infty} \inf I_{m,\beta}(v_n) \geq \int_{[v>\beta]}[\frac{1}{p}f(v)v - F(v)]. \tag{3.10}$$

From (3.9) and (3.10), we obtain

$$c_{m,0} < \lim_{n\to\infty} \inf I_{m,\beta}(v_n) + \frac{\delta}{8} = c_\beta + \frac{\delta}{8},$$

that is,

$$c_{m,0} < c_\beta + \frac{\delta}{8}.$$

From (4.18) of [3], we have

$$c_{m,0} < c_0 + \frac{\delta}{2} + \frac{\delta}{8} < c_0 + \delta.$$

On the other hand, by (3.1), we can obtain

$$c_{m,0} < c_{m,0} - \delta,$$

which is an absurd. Thus, the weak limit u is nontrivial.

Proof of Theorem 1.1. Once that the Lemma 3.1 and 3.2 were established, we can repeat the same arguments explored in the proof of Theorem 2.1 to conclude that u is a positive solution for $(P)_\beta$.

4 Conclusion

In this paper, by using variational methods, we studied the existence of positive solutions for quasilinear elliptic equation with discontinuous nonlinearity and nonlinear perturbations. Just as we know, when the nonlinear term is discontinuous and the function V is periodic, our first difficulty involving this class of problem is the fact that we can not use the classical variational methods, and it is necessary to use some results for Locally Lipschitz functional. Moreover, when the nonlinearity is continuous and satisfies some assumptions, the mountain pass level is equal to the minimum of the energy functional on Nehari Manifolds, which is a key point in a lot of papers. However, this property is not true for discontinuous nonlinearity. Hence, the arguments used in the above reference can not be repeated directly, and a careful analysis is necessary to get similar results to those found in [8,29,30]. What's more, the problem we have studied is one of the mathematical models occurring in the studies of the p-Laplacian equation, generalized reaction-diffusion theory, non-Newtonian fluid theory, and the turbulent flow of a gas in porous medium. In the non-Newtonian fluid theory, the quantity p is characteristic of the medium. Media with $p > 2$ are called dilatant fluids and those with $p < 2$ are called pseudoplastics. If $p = 2$, they are Newtonian fluids. When $p \neq 2$, the problem becomes more complicated since certain nice properties inherent to the case $p = 2$ seem to be lost or at least difficult to verify.

Competing Interests

The authors declare that no competing interests exist.

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