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The Multinomial Combinatorial Convolution Sum

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Abstract

In [1] and [3] we can find some formulas of binomial combinatorial convolution sums involving divisor functions. Starting from these formulas, we obtain a multinomial combinatorial convolution sum.

Keywords: Divisor functions; Combinatoric convolution sum

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1 Introduction

Let $\mathbb{N}$ denote the set of positive integers. Furthermore let $N, d \in \mathbb{N}$ and $s \in \mathbb{N} \cup \{0\}$. Throughout this paper, we define

$$\sigma_s(N) := \sum_{d|N} d^s, \quad \sigma_s^*(N) := \sum_{\substack{d|N \\ N/d \text{ odd}}} d^s,$$
$$(a)_N := (a)(a-1)\cdots(a-N+1).$$

We can deduce the property

$$\sigma_s^*(2N) = 2^s \sigma_s^*(N). \tag{1.1}$$

For example, we can see that

$$\sum_{m=1}^{N-1} \sigma_1^*(m)\sigma_1^*(N-m) = \frac{1}{4}\left(\sigma_3^*(N) - N\sigma_1^*(N)\right)$$

in [3] [Example 12.4]. We introduce very important Proposition 1.1 and Proposition 1.2, which is the starting point of our paper.

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Proposition 1.1. *(See [3] [Theorem 12.3]) Let* $k, N \in \mathbb{N}$*. Then*

$$\sum_{s=0}^{k-1}\binom{2k}{2s+1}\left(\sum_{m=1}^{N-1}\sigma_{2k-2s-1}(m)\sigma_{2s+1}(N-m)\right)$$
$$=\frac{2k+3}{4k+2}\sigma_{2k+1}(N)+\left(\frac{k}{6}-N\right)\sigma_{2k-1}(N)+\frac{1}{2k+1}\sum_{j=2}^{k}\binom{2k+1}{2j}B_{2j}\sigma_{2k+1-2j}(N).$$

Proposition 1.2. *(See [2] [Theorem 10])*
Let $l, N \in \mathbb{N}$ *with* $8|N$ *and define*

$$C := \Big\{(m_1, m_2, \cdots m_{11}, m_{12}) \in \mathbb{N}^{12} \big| \sum_{i=1}^{12} m_i = N, \quad 2|(m_1+m_2), \quad 2|(m_3+m_4),$$
$$2|(m_5+m_6), \quad 2|(m_7+m_8), \quad 2|(m_9+m_{10}), \quad 2|(m_{11}+m_{12}),$$
$$4|(m_1+m_2+m_3+m_4), \quad 4|(m_5+m_6+m_7+m_8),$$
$$8|(m_9+m_{10}+m_{11}+m_{12})\Big\}.$$

Then for odd positive integers a_i*, we have*

$$\sum_{\substack{\sum_{i=1}^{12} a_i=2l \\ a_i \text{ odd}}} (a_1+a_2)(a_3+a_4)(a_5+a_6)(a_7+a_8)(a_9+a_{10})(a_{11}+a_{12})$$
$$\times (a_1+a_2+a_3+a_4-3)(a_1+a_2+a_3+a_4-2)(a_5+a_6+a_7+a_8-2)$$
$$\times (a_9+a_{10}+a_{11}+a_{12}-2)\binom{2l}{a_1,a_2,a_3,a_4,a_5,a_6,a_7,a_8,a_9,a_{10},a_{11},a_{12}}$$
$$\times \sum_{C}\frac{(-1)^{m_1+m_3+m_5+m_7+m_9+m_{11}}}{(m_1+m_2)(m_3+m_4)(m_5+m_6)(m_7+m_8)(m_9+m_{10})(m_{11}+m_{12})}$$
$$\times \frac{(-1)^{\frac{m_1+m_2+m_3+m_4}{4}+\frac{m_1+m_2}{2}+\frac{m_5+m_6}{2}+\frac{m_9+m_{10}}{2}}}{(m_1+m_2+m_3+m_4)(m_5+m_6+m_7+m_8)(m_9+m_{10}+m_{11}+m_{12})}$$
$$\times \prod_{i=1}^{12}\sigma^*_{a_i}(m_i)$$
$$= 2^{-16}(2l)_{10}N\left\{\sigma^*_{2l-9}(N)-8N\sigma^*_{2l-11}(N)\right\}.$$

Similarly, getting an idea from Proposition 1.2, we obtain a multinomial combinatorial convolution sum as follows :

Theorem 1.1. *Let* $l, N \in \mathbb{N}$ *with* $16|N$ *and define*
D := $\big\{(m_1, m_2, \cdots m_{23}, m_{24}) \in \mathbb{N}^{24} \big| \sum_{i=1}^{24} m_i = N, \quad 2|(m_1+m_2), \quad 2|(m_3+m_4),$
$2|(m_5+m_6), \quad 2|(m_7+m_8), \quad 2|(m_9+m_{10}), \quad 2|(m_{11}+m_{12}),$
$2|(m_{13}+m_{14}), \quad 2|(m_{15}+m_{16}), \quad 2|(m_{17}+m_{18}), \quad 2|(m_{19}+m_{20}),$
$2|(m_{21}+m_{22}), \quad 2|(m_{23}+m_{24}), \quad 4|(m_1+m_2+m_3+m_4),$
$4|(m_5+m_6+m_7+m_8), \quad 4|(m_9+m_{10}+m_{11}+m_{12}),$
$4|(m_{13}+m_{14}+m_{15}+m_{16}), \quad 4|(m_{17}+m_{18}+m_{19}+m_{20}),$
$4|(m_{21}+m_{22}+m_{23}+m_{24}), \quad 8|(m_1+m_2+m_3+m_4+m_5+m_6+m_7+m_8),$

$8|(m_9+m_{10}+m_{11}+m_{12}+m_{13}+m_{14}+m_{15}+m_{16})$,
$16|(m_{17}+m_{18}+m_{19}+m_{20}+m_{21}+m_{22}+m_{23}+m_{24})\}$.

Then for odd positive integers a_i, we have

$$\begin{aligned}
&\sum_{\substack{\sum_{i=1}^{24} a_i=2l \\ a_i \text{ odd}}} (a_1+a_2)(a_3+a_4)(a_5+a_6)(a_7+a_8)(a_9+a_{10})(a_{11}+a_{12}) \\
&\quad \times (a_{13}+a_{14})(a_{15}+a_{16})(a_{17}+a_{18})(a_{19}+a_{20})(a_{21}+a_{22})(a_{23}+a_{24}) \\
&\quad \times (a_1+a_2+a_3+a_4-2)(a_5+a_6+a_7+a_8-2)(a_9+a_{10}+a_{11}+a_{12}-2) \\
&\quad \times (a_{13}+a_{14}+a_{15}+a_{16}-2)(a_{17}+a_{18}+a_{19}+a_{20}-2)(a_{21}+a_{22}+a_{23}+a_{24}-2) \\
&\quad \times (a_1+a_2+a_3+a_4+a_5+a_6+a_7+a_8-7) \\
&\quad \times (a_1+a_2+a_3+a_4+a_5+a_6+a_7+a_8-6) \\
&\quad \times (a_9+a_{10}+a_{11}+a_{12}+a_{13}+a_{14}+a_{15}+a_{16}-6) \\
&\quad \times (a_{17}+a_{18}+a_{19}+a_{20}+a_{21}+a_{22}+a_{23}+a_{24}-6) \\
&\quad \times \binom{2l}{a_1,a_2,a_3,a_4,a_5,a_6,a_7,a_8,a_9,a_{10},a_{11},a_{12},a_{13},a_{14},a_{15},a_{16},a_{17},a_{18},a_{19},a_{20},a_{21},a_{22},a_{23},a_{24}} \\
&\quad \times \sum_D (-1)^{m_1+m_3+m_5+m_7+m_9+m_{11}+m_{13}+m_{15}+m_{17}+m_{19}+m_{21}+m_{23}} \\
&\quad \times (-1)^{\frac{m_1+m_2+m_3+m_4+m_5+m_6+m_7+m_8}{8}+\frac{m_1+m_2+m_3+m_4}{4}} \\
&\quad \times (-1)^{\frac{m_9+m_{10}+m_{11}+m_{12}}{4}+\frac{m_{17}+m_{18}+m_{19}+m_{20}}{4}} \\
&\quad \times (-1)^{\frac{m_1+m_2+m_5+m_6+m_9+m_{10}+m_{13}+m_{14}+m_{17}+m_{18}+m_{21}+m_{22}}{2}} \\
&\quad \times \frac{1}{(m_1+m_2)(m_3+m_4)(m_5+m_6)(m_7+m_8)(m_9+m_{10})(m_{11}+m_{12})} \\
&\quad \times \frac{1}{(m_{13}+m_{14})(m_{15}+m_{16})(m_{17}+m_{18})(m_{19}+m_{20})(m_{21}+m_{22})(m_{23}+m_{24})} \\
&\quad \times \frac{1}{(m_1+m_2+m_3+m_4)(m_5+m_6+m_7+m_8)(m_9+m_{10}+m_{11}+m_{12})} \\
&\quad \times \frac{1}{(m_{13}+m_{14}+m_{15}+m_{16})(m_{17}+m_{18}+m_{19}+m_{20})(m_{21}+m_{22}+m_{23}+m_{24})} \\
&\quad \times \frac{1}{(m_1+m_2+m_3+m_4+m_5+m_6+m_7+m_8)} \\
&\quad \times \frac{1}{(m_9+m_{10}+m_{11}+m_{12}+m_{13}+m_{14}+m_{15}+m_{16})} \\
&\quad \times \frac{1}{(m_{17}+m_{18}+m_{19}+m_{20}+m_{21}+m_{22}+m_{23}+m_{24})} \prod_{i=1}^{24} \sigma_{a_i}^*(m_i) \\
&= 2^{-29}(2l)_{22} N\left\{\sigma_{2l-21}^*(N)-16N\sigma_{2l-23}^*(N)\right\}.
\end{aligned} \tag{1.2}$$

2 Proof of Theorem 1.1

It is similar to Proposition 1.2. From $16|N$, we have $8|\frac{N}{2}$. Let $a_1+a_2:=2k_1$, $a_3+a_4:=2k_2$, $a_5+a_6:=2k_3$, $a_7+a_8:=2k_4$, $a_9+a_{10}:=2k_5$, $a_{11}+a_{12}:=2k_6$, $a_{13}+a_{14}:=2k_7$, $a_{15}+a_{16}:=2k_8$,

$a_{17} + a_{18} := 2k_9$, $a_{19} + a_{20} := 2k_{10}$, $a_{21} + a_{22} := 2k_{11}$, $a_{23} + a_{24} := 2k_{12}$. And the condition

$$\begin{aligned} D := \Bigg\{ &(m_1, m_2, \cdots m_{23}, m_{24}) \in \mathbb{N}^{24} | \sum_{i=1}^{24} m_i = N, \quad 2|(m_1 + m_2), \quad 2|(m_3 + m_4), \\ &2|(m_5 + m_6), \quad 2|(m_7 + m_8), \quad 2|(m_9 + m_{10}), \quad 2|(m_{11} + m_{12}), \\ &2|(m_{13} + m_{14}), \quad 2|(m_{15} + m_{16}), \quad 2|(m_{17} + m_{18}), \quad 2|(m_{19} + m_{20}), \\ &2|(m_{21} + m_{22}), \quad 2|(m_{23} + m_{24}), \quad 4|(m_1 + m_2 + m_3 + m_4), \\ &4|(m_5 + m_6 + m_7 + m_8), \quad 4|(m_9 + m_{10} + m_{11} + m_{12}), \\ &4|(m_{13} + m_{14} + m_{15} + m_{16}), \quad 4|(m_{17} + m_{18} + m_{19} + m_{20}), \\ &4|(m_{21} + m_{22} + m_{23} + m_{24}), \quad 8|(m_1 + m_2 + m_3 + m_4 + m_5 + m_6 + m_7 + m_8), \\ &8|(m_9 + m_{10} + m_{11} + m_{12} + m_{13} + m_{14} + m_{15} + m_{16}), \\ &16|(m_{17} + m_{18} + m_{19} + m_{20} + m_{21} + m_{22} + m_{23} + m_{24})\} \end{aligned}$$

shows us that

$$\begin{aligned} m_1 + m_2 = 2N_1, \quad m_3 + m_4 = 2N_2, \quad m_5 + m_6 = 2N_3, \\ m_7 + m_8 = 2N_4, \quad m_9 + m_{10} = 2N_5, \quad m_{11} + m_{12} = 2N_6, \\ m_{13} + m_{14} = 2N_7, \quad m_{15} + m_{16} = 2N_8, \quad m_{17} + m_{18} = 2N_9, \\ m_{19} + m_{20} = 2N_{10}, \quad m_{21} + m_{22} = 2N_{11}, \quad m_{23} + m_{24} = 2N_{12} \end{aligned}$$

and these facts and the leaving conditions of D lead that

$$\begin{aligned} E := \Bigg\{ &(N_1, N_2, \cdots N_{11}, N_{12}) \in \mathbb{N}^{12} | \sum_{i=1}^{12} N_i = \frac{N}{2}, \quad 2|(N_1 + N_2), \quad 2|(N_3 + N_4), \\ &2|(N_5 + N_6), \quad 2|(N_7 + N_8), \quad 2|(N_9 + N_{10}), \quad 2|(N_{11} + N_{12}), \\ &4|(N_1 + N_2 + N_3 + N_4), \quad 4|(N_5 + N_6 + N_7 + N_8), \\ &8|(N_9 + N_{10} + N_{11} + N_{12})\}. \end{aligned} \tag{2.1}$$

Considering (2.1), we can write (1.2) as follows :

$$
\begin{aligned}
&\sum_{\substack{\sum_{i=1}^{24} a_i = 2l \\ a_i \text{ odd}}} (a_1+a_2)(a_3+a_4)(a_5+a_6)(a_7+a_8)(a_9+a_{10})(a_{11}+a_{12}) \\
&\times (a_{13}+a_{14})(a_{15}+a_{16})(a_{17}+a_{18})(a_{19}+a_{20})(a_{21}+a_{22})(a_{23}+a_{24}) \\
&\times (a_1+a_2+a_3+a_4-2)(a_5+a_6+a_7+a_8-2)(a_9+a_{10}+a_{11}+a_{12}-2) \\
&\times (a_{13}+a_{14}+a_{15}+a_{16}-2)(a_{17}+a_{18}+a_{19}+a_{20}-2)(a_{21}+a_{22}+a_{23}+a_{24}-2) \\
&\times (a_1+a_2+a_3+a_4+a_5+a_6+a_7+a_8-7) \\
&\times (a_1+a_2+a_3+a_4+a_5+a_6+a_7+a_8-6) \\
&\times (a_9+a_{10}+a_{11}+a_{12}+a_{13}+a_{14}+a_{15}+a_{16}-6) \\
&\times (a_{17}+a_{18}+a_{19}+a_{20}+a_{21}+a_{22}+a_{23}+a_{24}-6) \\
&\times \frac{2l!}{(a_1+a_2)!(a_3+a_4)!(a_5+a_6)!(a_7+a_8)!(a_9+a_{10})!(a_{11}+a_{12})!} \\
&\times \frac{1}{(a_{13}+a_{14})!(a_{15}+a_{16})!(a_{17}+a_{18})!(a_{19}+a_{20})!(a_{21}+a_{22})!(a_{23}+a_{24})!} \\
&\times \frac{(a_1+a_2)!}{a_1!a_2!}\cdot\frac{(a_3+a_4)!}{a_3!a_4!}\cdot\frac{(a_5+a_6)!}{a_5!a_6!}\cdot\frac{(a_7+a_8)!}{a_7!a_8!}\cdot\frac{(a_9+a_{10})!}{a_9!a_{10}!}\cdot\frac{(a_{11}+a_{12})!}{a_{11}!a_{12}!} \\
&\times \frac{(a_{13}+a_{14})!}{a_{13}!a_{14}!}\cdot\frac{(a_{15}+a_{16})!}{a_{15}!a_{16}!}\cdot\frac{(a_{17}+a_{18})!}{a_{17}!a_{18}!}\cdot\frac{(a_{19}+a_{20})!}{a_{19}!a_{20}!}\cdot\frac{(a_{21}+a_{22})!}{a_{21}!a_{22}!}\cdot\frac{(a_{23}+a_{24})!}{a_{23}!a_{24}!} \\
&\times \sum_D \frac{(-1)^{m_1+1}}{(m_1+m_2)}\cdot\frac{(-1)^{m_3+1}}{(m_3+m_4)}\cdot\frac{(-1)^{m_5+1}}{(m_5+m_6)}\cdot\frac{(-1)^{m_7+1}}{(m_7+m_8)}\cdot\frac{(-1)^{m_9+1}}{(m_9+m_{10})}\cdot\frac{(-1)^{m_{11}+1}}{(m_{11}+m_{12})} \\
&\times \frac{(-1)^{m_{13}+1}}{(m_{13}+m_{14})}\cdot\frac{(-1)^{m_{15}+1}}{(m_{15}+m_{16})}\cdot\frac{(-1)^{m_{17}+1}}{(m_{17}+m_{18})}\cdot\frac{(-1)^{m_{19}+1}}{(m_{19}+m_{20})}\cdot\frac{(-1)^{m_{21}+1}}{(m_{21}+m_{22})}\cdot\frac{(-1)^{m_{23}+1}}{m_{23}+m_{24}} \\
&\times (-1)^{\frac{m_1+m_2+m_3+m_4+m_5+m_6+m_7+m_8}{8}+\frac{m_1+m_2+m_3+m_4}{4}} \\
&\times (-1)^{\frac{m_9+m_{10}+m_{11}+m_{12}}{4}+\frac{m_{17}+m_{18}+m_{19}+m_{20}}{4}} \\
&\times (-1)^{\frac{m_1+m_2+m_5+m_6+m_9+m_{10}+m_{13}+m_{14}+m_{17}+m_{18}+m_{21}+m_{22}}{2}} \\
&\times \frac{1}{(m_1+m_2+m_3+m_4)(m_5+m_6+m_7+m_8)(m_9+m_{10}+m_{11}+m_{12})} \\
&\times \frac{1}{(m_{13}+m_{14}+m_{15}+m_{16})(m_{17}+m_{18}+m_{19}+m_{20})(m_{21}+m_{22}+m_{23}+m_{24})} \\
&\times \frac{1}{(m_1+m_2+m_3+m_4+m_5+m_6+m_7+m_8)} \\
&\times \frac{1}{(m_9+m_{10}+m_{11}+m_{12}+m_{13}+m_{14}+m_{15}+m_{16})} \\
&\times \frac{1}{(m_{17}+m_{18}+m_{19}+m_{20}+m_{21}+m_{22}+m_{23}+m_{24})}\prod_{i=1}^{24}\sigma^*_{a_i}(m_i).
\end{aligned} \tag{2.2}
$$

Then applying the [1], [(16)] as the following form

$$
\sum_{s=0}^{k-1}\binom{2k}{2s+1}\sum_{m=1}^{2N-1}\frac{(-1)^{m+1}\sigma^*_{2k-2s-1}(m)\sigma^*_{2s+1}(2N-m)}{N} = \sigma^*_{2k-1}(2N)
$$

into (2.2), we obtain

$$
\begin{aligned}
&\sum_{\substack{\sum_{i=1}^{12} k_i = l \\ k_i \geq 1}} (2(k_1+k_2+k_3+k_4-2)-3)\,(2(k_1+k_2+k_3+k_4-2)-2) \\
&\quad \times (2(k_5+k_6+k_7+k_8-2)-2)\,(2(k_9+k_{10}+k_{11}+k_{12}-2)-2) \\
&\quad \times 2(k_1+k_2-1)\cdot 2(k_3+k_4-1)\cdot 2(k_5+k_6-1)\cdot 2(k_7+k_8-1) \\
&\quad \times 2(k_9+k_{10}-1)\cdot 2(k_{11}+k_{12}-1) \\
&\quad \times 2k_1\cdot 2k_2\cdot 2k_3\cdot 2k_4\cdot 2k_5\cdot 2k_6\cdot 2k_7\cdot 2k_8\cdot 2k_9\cdot 2k_{10}\cdot 2k_{11}\cdot 2k_{12} \\
&\quad \times \frac{2l!}{2k_1!\cdot 2k_2!\cdot 2k_3!\cdot 2k_4!\cdot 2k_5!\cdot 2k_6!\cdot 2k_7!\cdot 2k_8!\cdot 2k_9!\cdot 2k_{10}!\cdot 2k_{11}!\cdot 2k_{12}!} \\
&\quad \times \frac{1}{2^{21}}\sum_E \frac{(-1)^{N_1+N_3+N_5+N_7+N_9+N_{11}}}{(N_1+N_2)(N_3+N_4)(N_5+N_6)(N_7+N_8)(N_9+N_{10})(N_{11}+N_{12})} \\
&\quad \times \frac{(-1)^{\frac{N_1+N_2+N_3+N_4}{4}+\frac{N_1+N_2}{2}+\frac{N_5+N_6}{2}+\frac{N_9+N_{10}}{2}}}{(N_1+N_2+N_3+N_4)(N_5+N_6+N_7+N_8)(N_9+N_{10}+N_{11}+N_{12})} \\
&\quad \times \prod_{i=1}^{12} \sigma^*_{a_{2k_i-1}}(2N_i) \\
&= \frac{1}{2^{21}}(2l)(2l-1)(2l-2)(2l-3)(2l-4)(2l-5)(2l-6) \\
&\quad \times (2l-7)(2l-8)(2l-9)(2l-10)(2l-11) \\
&\quad \times \sum_{\substack{\sum_{i=1}^{12} k_i = l \\ k_i \geq 1}} 2(k_1+k_2-1)\cdot 2(k_3+k_4-1)\cdot 2(k_5+k_6-1) \\
&\quad \times 2(k_7+k_8-1)\cdot 2(k_9+k_{10}-1)\cdot 2(k_{11}+k_{12}-1) \\
&\quad \times (2(k_1+k_2+k_3+k_4-2)-3)\,(2(k_1+k_2+k_3+k_4-2)-2) \\
&\quad \times (2(k_5+k_6+k_7+k_8-2)-2)\,(2(k_9+k_{10}+k_{11}+k_{12}-2)-2) \\
&\quad \times \binom{2(l-6)}{2k_1-1, 2k_2-1, 2k_3-1, 2k_4-1, ..., 2k_{11}-1, 2k_{12}-1} \\
&\quad \times \sum_E \frac{(-1)^{N_1+N_3+N_5+N_7+N_9+N_{11}}}{(N_1+N_2)(N_3+N_4)(N_5+N_6)(N_7+N_8)(N_9+N_{10})(N_{11}+N_{12})} \\
&\quad \times \frac{(-1)^{\frac{N_1+N_2+N_3+N_4}{4}+\frac{N_1+N_2}{2}+\frac{N_5+N_6}{2}+\frac{N_9+N_{10}}{2}}}{(N_1+N_2+N_3+N_4)(N_5+N_6+N_7+N_8)(N_9+N_{10}+N_{11}+N_{12})} \\
&\quad \times 2^{2l-12}\prod_{i=1}^{12} \sigma^*_{a_{2k_i-1}}(N_i),
\end{aligned} \tag{2.3}
$$

where we use (1.1), and the facts $\sum_{i=1}^{12} k_i = l$ and $\sum_{i=1}^{12} a^{2k_i-1} = a^{2l-12}$. So we insert $l \to l-6$ and $N \to \frac{N}{2}$ in Proposition 1.2 and we apply this induced formula into (2.3), thus (2.3) becomes

$$
\begin{aligned}
&2^{-21}(2l)(2l-1)(2l-2)(2l-3)(2l-4)(2l-5)(2l-6)\\
&\quad\times(2l-7)(2l-8)(2l-9)(2l-10)(2l-11)\\
&\quad\times 2^{-16}\cdot 2(l-6)\,(2(l-6)-1)\,(2(l-6)-2)\,(2(l-6)-3)\\
&\quad\times(2(l-6)-4)\,(2(l-6)-5)\,(2(l-6)-6)\\
&\quad\times(2(l-6)-7)\,(2(l-6)-8)\,(2(l-6)-9)\\
&\quad\times 2^{2l-12}\cdot\frac{N}{2}\left\{\sigma^*_{2(l-6)-9}(\frac{N}{2})-8\cdot\frac{N}{2}\sigma^*_{2(l-6)-11}(\frac{N}{2})\right\}\\
&=2^{-29}(2l)_{22}N\left\{\sigma^*_{2l-21}(N)-16N\sigma^*_{2l-23}(N)\right\},
\end{aligned}
$$

where we use (1.1) again and the definition of $(a)_N$.

3 Conclusions

In general we search the formula of the convolution sum as $\sum_{m=1}^{n-1}\sigma_i(m)\sigma_j(n-m)$ for $i,j\in\mathbb{N}$. But in this paper, we find the formula of the sum of those convolution sums as

$$\sum_{\substack{\sum_{i=1}^{24} a_i=2l\\ a_i\ \text{odd}}}\prod_{i=1}^{24}\sigma^*_{a_i}(m_i)$$

for $a_i, m_i\in\mathbb{N}$ adding some special conditions.

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Competing Interests

The author declares that no competing interests exist.

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