American Journal of Computational Mathematics, 2012, 2, 21-26
http://dx.doi.org/10.4236/ajcm.2012.21003 Published Online March 2012 (http://www.SciRP.org/journal/ajcm)

Computation of the Smith Form for Multivariate Polynomial Matrices Using Maple

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Received October 13, 2011; revised December 2, 2011; accepted December 15, 2011

ABSTRACT

In this paper we show how the transformations associated with the reduction to the Smith form of some classes of multivariate polynomial matrices are computed. Using a Maple implementation of a constructive version of the Quillen-Suslin Theorem, we present two algorithms for the reduction to a particular Smith form often associated with the simplification of linear systems of multidimensional equations.

Keywords: Smith Form; Unimodular; Equivalence; Quillen-Suslin Theorem; Maple

1. Introduction

Matrices whose elements are polynomials in more than one indeterminate have been studied by many authors. Such matrices arise in the mathematical treatment of the so-called multidimensional systems which can be considered as extensions of the ordinary differential or difference systems. These include delay-differential systems and partial differential systems. In particular the Smith normal form of a matrix plays an important role in many areas of mathematics such as the polynomial approach in control theory see for example Rosenbrock [1] and Kailath [2]. The problem of reducing an univariate polynomial matrix to its Smith form is well understood and the relevant algorithm is already implemented in most computer algebra systems. For the multivariate case, however the problem is still open. Despite that some necessary and/or sufficient conditions for the reduction of a matrix to its Smith form have been given in the literature, no algorithm has been given to show how the transformations involved in the reduction are actually computed. These transformations are important if the solutions of the reduced system are to be expressed in terms of the original variables. So far the computations associated with these conditions have been difficult if not impossible to carry out. Recently however, some progress has been made in symbolic computation in particular a QuillenSuslin Maple package [3] has been developed. This package which is a maple implementa- tion of the Quillen-Suslin theorem provides an algorithm which computes a basis of a free module over a polynomial ring. In terms of matrices, this algorithm completes a unimodular rectangular matrix to an invertible matrix over the given polynomial ring with rational or integer coefficients (for more details see http://wwwb.math.rwth-aachen.de/QuillenSuslin/). In this paper, we show how this package can be used to compute the Smith form and the associated transformation for some classes of multivariate polynomial matrices. The classes of matrices considered can be regarded as those associated with linear determined systems of multidimensional equations which can be reduced to a single equation, thereby simplifying the analysis of such systems. The transformation used to obtain the Smith form is that of unimodular equivalence which will be defined later. First we need to introduce some definitions and a theorem which play a key role in this paper.

Definition 1 *Let* D *be a ring. The general linear group* $GL_p(D)$ *is defined by*

$$GL_p(D)=\left\{M\in D^{p\times p}\,\middle|\,\exists N\in D^{p\times p}:MN=NM=I_p\right\}$$

An element $M\in GL_p(D)$ *is called a unimodular matrix. In the case where* $D=K[x_1,\cdots,x_n]$, *a polynomial ring in the indeterminates* $x_1,\cdots,x_n$ *and coefficients in* K, *the matrix* M *is unimodular if and only if the determinant of* M *is invertible in* D, *i.e., is a non-zero element of* K.

In the case when a matrix is rectangular we introduce the following concept of primeness.

Definition 2 *A matrix* $R\in D^{q\times p}$ *with* $p>q$ *is said to be zero-left-prime* (ZLP) *if it admits a right-inverse over* D. *i.e., if there exists* $\tilde{R}\in D^{p\times q}$ *such that* $R\tilde{R}=I_q$.

Similarly zero-right-primeness (ZRP) can be defined by transposition for matrices where $p<q$.

We now state the famous Quillen-Suslin Theorem.

This theorem is at the centre of the proofs of the results presented in this paper and its Maple implementation is used in the computation of the transformations.

Theorem 1 (Quillen-Suslin [4,5]) *Let K be a principal ideal domain and $D = K(x_1,\cdots,x_n)$ and let $R \in D^{q\times p}$ be a matrix which admits a right-inverse $\tilde{R} \in D^{p\times q}$, i.e., $R\tilde{R} = I_q$. Then there exists a unimodular matrix $N \in GL_p(D)$ such that*

$$R\tilde{N} = \begin{pmatrix} I_q & 0 \end{pmatrix} \quad (1)$$

2. Equivalence to the Smith Form

The Smith form S of a $p\times q$ matrix T with elements in a ring D is usually the result of an equivalence transformation, *i.e.* a transformation of the form

$$S = MTN \quad (2)$$

where $M \in GL_p(D)$ and $M \in GL_q(D)$. The resulting Smith form *S* is given by the diagonal matrix formed by the invariant polynomials. The non-zero invariant polynomials $\gamma_1,\cdots,\gamma_r$ are defined by $\gamma_i = \alpha_i/\alpha_{i-1}$ with $\alpha_0 = 1$, r is the rank of T and α_i is the gcd of the $i\times i$ minors of T.

It is a fact that it is not always possible to reduce a matrix to its Smith form by an equivalence of the type (2). In order to show that any matrix can be brought by an equivalence transformation to its Smith form, it is usually required that D is a principal ideal or a Euclidean domain. In the case when $D = \mathbb{R}[z_1,\cdots,z_n]$, the matrices may be treated as having elements in one of the indeterminates with coefficients that are rational forms in the other indeterminates e.g. $\mathbb{R}(z_1,\cdots,z_{n-1})[z_n]$. This approach which involves a renormalization step has the disadvantage of yielding Smith forms which are not unique, see the work of Frost and Storey [6] and Morf *et al.* [7]. Conditions under which a matrix with elements in $\mathbb{R}[z_1,\cdots,z_n]$ is equivalent to its Smith form have been given by a number of authors. For the ring $\mathbb{R}[z_1,z_2]$, Lee and Zak [8] proposed a necessary and sufficient condition in terms of the existence of solutions to certain polynomial equations. Frost and Boudellioua [9] gave a necessary and sufficient condition for a class of multivariate polynomial matrices in terms of the existence of a polynomial vector. Lin *et al.* [10] proposed a sufficient condition for a class of matrices whose determinant is linear in one of the indeterminates.

Theorem 2 (Boudellioua and Quadrat [11]) *Let $D = \mathbb{R}[z_1,\cdots,z_n]$, a matrix $T \in D^{p\times p}$ is equivalent to the Smith form*:

$$S = \begin{pmatrix} I_{p-1} & 0 \\ 0 & |T| \end{pmatrix} \quad (3)$$

if and only if there exists a ZRP vector $U \in D^p$ such that the matrix $\begin{pmatrix} T & U \end{pmatrix}$ is ZLP.

This is a particular type of Smith form where all the invariant polynomials are equal to 1 except the last one which is given by the determinant of the matrix. This form is important for simplification considerations, *i.e.* a system whose matrix is equivalent to the Smith form (3) is equivalent to a single equation in one unknown. Thus making it easier to analyse such a system either analyticcally or numerically. In order to express the conclusions and solutions made about the reduced system in terms of the original system, one has to compute the transformations (2) connecting the original system matrix to the Smith form. Suppose that such a vector is obtained then the transformations M and N that reduce T to the Smith form (3) can be obtained via the following Maple algorithm.

2.1. Algorithm 1

- *Declare the path where the required Maple libraries such as QuillenSuslin are stored and load the packages LinearAlgebra and QuillenSuslin.*
- *Declare the ring over which the matrix is defined by declaring the indeterminates and the field of coefficients.*
- *Enter the elements of the matrices T and U.*
- *Test the zero-primeness of the matrix $\begin{pmatrix} T & U \end{pmatrix}$ using the function IsUnimod.*
- *Use the function QSAlgorithm to compute the square unimodular matrix M_0 such that $U^T M_0^T = \begin{pmatrix} I_p & 0 \end{pmatrix}$.*
- *Interchange the first and last column of M_0^T and transpose to obtain the matrix M_1.*
- *Extract the matrices $K_i, i = 1\cdots 4$ and $L_j, j = 1\cdots 2$ from M_1 and T respectively.*
- *Compute the matrix $T_1 = K_1L_1 + K_2L_2$, where* $M_1 = \begin{pmatrix} K_1 & K_2 \\ K_3 & K_4 \end{pmatrix}$ *and* $T = \begin{pmatrix} L_1 \\ L_2 \end{pmatrix}$.
- *Use the function QSAlgorithm to obtain the square unimodular matrix N such that $T_1N = \begin{pmatrix} I_{p-1} & 0 \end{pmatrix}$.*
- *Extract the submatrix R formed from the first row and the first $p-1$ columns of the matrix $K_3L_1 + K_4L_2$.*
- *Construct the matrix* $M = \begin{pmatrix} K_1 & K_2 \\ -RK_1 + K_3 & -RK_2 + K_4 \end{pmatrix}$.
- *Check that the Smith form $S = MTN$.*

Another interesting result in the form of a sufficient condition for the reduction of class of linear multidimensional systems was given by Lin *et al.* [10]. This is the class of systems where the determinant of the system matrix is linear in one of the indeterminates. Such systems are also equivalent to a single equation. The result is given for the case when the determinant is linear in z_1 but it is equally valid for any indeterminate $z_i, i = 1,\cdots,n.$

Theorem 3 (Lin *et al.* [10]) *Let* $D = \mathbb{R}[z_1,\cdots,z_n]$

and $T \in D^{p\times p}$, *then if* $|T| = z_1 - f(z_2,\cdots,z_n)$, T *is equivalent to the Smith form:*

$$S = \begin{pmatrix} I_{p-1} & 0 \\ 0 & |T| \end{pmatrix}. \quad (4)$$

In the following we give an algorithm to find the unimodular matrices *M* and *N* that reduce the matrix *T* to the Smith form *S*.

2.2. Algorithm 2

- *Declare the path where the required Maple libraries such as QuillenSuslin are stored and load the packages LinearAlgebra and QuillenSuslin.*
- *Declare the ring* D *over which the matrix is defined by declaring the indeterminates and the field of coefficients.*
- *Enter the elements of the matrix* T.
- *Check that the determinant of* T *is linear in one of the indeterminates, say* z_i.
- *Compute* $f = z_i - |T|$.
- *Substitute* $z_i = f$ *in* T *to obtain* $\bar{T}$.
- *Compute a* ZRP *vector* $w \in \mathbb{R}^p[\cdots, z_{i-1}, z_{i+1}, \cdots]$ *such that* $\bar{T}w = 0$ *using the function SyzygyModule.*
- *Compute a matrix* $N \in GL_p(D)$ *with last column given by* w *using the function CompleteMatrix.*
- *Compute the matrix* $M \in GL_p(D)$ *given by*

  $M = SN^{-1}T^{-1}$, *where* $S = \begin{pmatrix} I_{p-1} & 0 \\ 0 & |T| \end{pmatrix}$.
- *Check that the Smith form* $S = MTN$.

Example 1 (Frost and Boudellioua [9])

libname:="C:/Involutive", "C:/QuillenSuslin", libname:

with (LinearAlgebra): with (QuillenSuslin):

Consider again the 3×3 matrix T in Example 1 over the polynomial ring $D = \mathbb{R}[s,z]$.

var := [*s*, *z*];

$$\text{var} := [s,z]$$

T := Matrix([[2*s*z^2 + z^3 + z^2 + 1,s*z^2 – s*z + s, 2*s*z + z^2], [2*s*z + z^2 + z, s*z – s, 2*s + z], [2*s^2*z + s*z^2 + s*z + z, s^2*z – s^2 – 1, 2*s^2 + s*z + 1]]); det_T := Determinant(T);

$$T := \begin{pmatrix} 2sz^2+z^3+z^2+1 & sz^2-sz+s & 2sz+z^2 \\ 2sz+z^2+z & sz-s & 2s+z \\ 2s^2z+sz^2+sz+z & s^2z-s^2-1 & 2s^2+sz+1 \end{pmatrix}$$

$$\det\_T := s+z$$

p := RowDimension(T);

$$p := 3$$

Now consider the column vector U satisfying the condition in Theorem 3,

U := Matrix([[z], [1], [s]]);

$$U := \begin{pmatrix} z \\ 1 \\ s \end{pmatrix}$$

UT := Transpose(U);

$$UT = \begin{pmatrix} z & 1 & s \end{pmatrix}$$

IsUnimod(UT, var, true);

true

TU := Matrix([T,U]);

$$TU := \begin{pmatrix} 2sz^2+z^3+z^2+1 & sz^2-sz+s & 2sz+z^2 & z \\ 2sz+z^2+z & sz-s & 2s+z & 1 \\ 2s^2z+sz^2+sz+z & s^2z-s^2-1 & 2s^2+sz+1 & s \end{pmatrix}$$

Let us check if the the matrix $\begin{pmatrix} T & U \end{pmatrix}$ is unimodular

IsUnimod (TU, var, true);

true

Applying the QSAlgorithm procedure to the row UT, we then obtain

MT := QSAlgorithm(UT, var, true);

$$MT := \begin{pmatrix} 0 & 1 & 0 \\ 1 & -z & -s \\ 0 & 0 & 1 \end{pmatrix}$$

UTMT := simplify(Matrix(UT). MT);

$$UTMT := \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$$

Now we can check that UT is the first row of the inverse of MT:

MT_inv := CompleteMatrix(UT, var, true);

$$MT\_inv := \begin{pmatrix} z & 1 & s \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

i.e., the matrix MT_inv is a completion of UT to an invertible matrix over D.

M1 := Transpose(ColumnOperation (Matrix(MT), [1, p]));

$$M1 := \begin{pmatrix} 0 & -s & 1 \\ 1 & -z & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

The matrix $M1 \equiv \begin{pmatrix} K1 & K2 \\ K3 & K4 \end{pmatrix}$ and $T \equiv \begin{pmatrix} L1 & L2 \end{pmatrix}^T$, where

K1 := DeleteColumn(DeleteRow(M1, p), p); K2 := DeleteColumn(DeleteRow(M1, p), 1⋯p – 1); K3 :=

DeleteColumn(DeleteRow(M1, 1⋯p – 1), p); K4 := DeleteColumn(DeleteRow(M1, 1⋯p – 1), 1⋯p – 1);

$$K1 := \begin{pmatrix} 0 & -s \\ 1 & -z \end{pmatrix}$$

$$K2 := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$K3 := \begin{pmatrix} 0 & 1 \end{pmatrix}$$

$$K4 := \begin{pmatrix} 0 \end{pmatrix}$$

L1 := DeleteRow(T, p); L2: = DeleteRow(T, 1⋯p – 1);

$$L1 := \begin{pmatrix} 2sz^2+z^3+z^2+1 & sz^2-sz+s & 2sz+z^2 \\ 2sz+z^2+z & sz-s & 2s+z \end{pmatrix}$$

$$L2 := \begin{pmatrix} 2s^2z+sz^2+sz+z & s^2z-s^2-1 & 2s^2+sz+1 \end{pmatrix}$$

T1 := simplify(K1.L1 + K2.L2);

$$T1 := \begin{pmatrix} z & -1 & 1 \\ 1 & s & 0 \end{pmatrix}$$

N := QSAlgorithm(T1, var, true);

$$N := \begin{pmatrix} 0 & 1 & -s \\ 0 & 0 & 1 \\ 1 & -z & 1+sz \end{pmatrix}$$

R := simplify(SubMatrix((K3.L1 + K4.L2). N, 1..1, 1..p – 1));

$$R := \begin{pmatrix} 2s+z & z \end{pmatrix}$$

M := Matrix([[K1, K2], [–R.K1 + K3, –R.K2 + K4]]);

$$M := \begin{pmatrix} 0 & -s & 1 \\ 1 & -z & 0 \\ -z & (2s+z)s+z^2+1 & -2s-z \end{pmatrix}$$

S := simplify(M.T.N);

$$S := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & s+z \end{pmatrix}$$

Example 2 (Frost and Boudellioua [9])

libname := “C: /Involutive”, “C: /QuillenSuslin”, libname:

with (LinearAlgebra): with (QuillenSuslin):

In the QuillenSuslin package all computations are performed over a commutative polynomial ring with rational coefficients if the last parameter *param* is set to *true* and with integer coefficients, if the parameter is set to *false*.

param := true:

The variables s and z of the polynomial ring $D = \mathbb{R}[s,z]$ must be declared by setting:

var := [s,z];

$$\text{var} := [s,z]$$

Now consider the 3×3 matrix $T(s,z)$ over the polynomial ring D.

T := Matrix([[2*s*z^2 + z^3 + z^2 + 1, s*z^2-s*z + s, 2*s*z + z^2],\ [2*s^2*z + s*z^2 + s*z + z, s^2*z – s^2 – 1, 2*s^2 + s*z + 1]]); det_T := Determinant(T);

$$T := \begin{pmatrix} 2sz^2+z^3+z^2+1 & sz^2-sz+s & 2sz+z^2 \\ 2sz+z^2+z & sz-s & 2s+z \\ 2s^2z+sz^2+sz+z & s^2z-s^2-1 & 2s^2+sz+1 \end{pmatrix}$$

$$\text{det\_}T := s+z$$

Clearly the determinant of $T(s,z)$ satisfies the condition in Theorem 3.

p := RowDimension(T);

$$p := 3$$

f := s-det_T;

$$f := -z$$

Evaluating $T(s,z)$ at $s=-z$, *i.e.* computing $T(-z,z) \equiv P(z)$,

P := subs(s = f,T);

$$P := \begin{pmatrix} -z^3+z^2+1 & -z^3+z^2-z & -z^2 \\ -z^2+z & -z^2+z & -z \\ z^3-z^2+z & z^3-z^2-1 & z^2+1 \end{pmatrix}$$

To find a unimodular column vector w satisfying $Pw=0$ we need to find the row vector defining the syzygy module of P over D.

w := Transpose(Matrix(SyzygyModule(Transpose(P), var)));

$$w := \begin{pmatrix} -z \\ -1 \\ z^2-1 \end{pmatrix}$$

Testing that the vector w is unimodular and that $Pw=0$.

IsUnimod (w,var,param);

true

Now compute a matrix $N \in GL_3(\mathbb{R}[z])$ with last column given by w

N := ColumnOperation (Transpose(CompleteMatrix (Transpose(w),\
var,false)),[1,p]);

$$N := \begin{pmatrix} 0 & 1 & -z \\ 0 & 0 & -1 \\ 1 & 0 & z^2-1 \end{pmatrix}$$

PN := simplify(P.N);

$$PN := \begin{pmatrix} -z^2 & -z^3+z^2+1 & 0 \\ -z & -z^2+z & 0 \\ z^2+1 & z^3-z^2+z & 0 \end{pmatrix}$$

Computing the matrix $N \in GL_3(\mathbb{R}[z])$ such that $MB = \begin{pmatrix} I_2 \\ 0 \end{pmatrix}$.

M := Transpose (QSAlgorithm(Transpose(B), var, true));

$$M := \begin{pmatrix} z^3-z^2-z & -2(z^3-z^2-1)z & -z^3+z^2+1 \\ -z^2+1 & 2z^3 & z^2 \\ -z & 2z^2+1 & z \end{pmatrix}$$

TN := simplify(T.N);

$$TN := \begin{pmatrix} 2sz+z^2 & 2sz^2+z^3+z^2+1 & -z^3-z-sz^2-sz-s-z^2 \\ 2s+z & 2sz+z^2+z & -z^2-sz-s-z \\ 2s^2+sz+1 & 2s^2z+sz^2+sz+z & -sz^2-s^2z-s^2-sz \end{pmatrix}$$

S := Matrix([[IdentityMatrix(p – 1), ZeroMatrix(p – 1,1)], \[ZeroMatrix(1, p – 1), det_T]]),

$$S := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & s+z \end{pmatrix}$$

M := simplify(S.MatrixInverse(N). MatrixInverse(T)); “N” = N;

$$M := \begin{pmatrix} -z^2-z & 2s^2z+sz^2+z^3+z^2+z-s & -z^2+1-2sz \\ z+1 & -2s^2-sz-z^2-z-1 & 2s+z \\ z & -2s^2-sz-z^2-1 & 2s+z \end{pmatrix}$$

$$N = \begin{pmatrix} 0 & 1 & -z \\ 0 & 0 & -1 \\ 1 & 0 & z^2-1 \end{pmatrix}$$

IsUnimod(M,var,true);

true

SS := simplify(M.T.N);

$$SS := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & s+z \end{pmatrix}$$

3. Conclusion

In this paper, we have shown that the recent implementation in Maple of a constructive version of the Quillen-Suslin Theorem can be used effectively to compute the Smith form and the associated unimodular transformations for a class of multivariate polynomial matrices. The classes of matrices considered are those arising from multidimensional systems amenable to be simplified to a single equation in one unknown. The case of underdetermined systems can also be treated in a similar fashion.

4. Acknowledgements

The author wishes to express his thanks to Sultan Qaboos University (Oman) for their support in carrying out this research and Dr. Alban Quadrat and Anna Fabianska for their help in providing the author with a copy of the QuillenSuslin Maple package.

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