Unit Monsef Distribution with Regression Model

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Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

Original Research Article

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Abstract

The aim of this paper is introduced a new distribution with one parameter called unit Monsef distribution (UMD) with it is regression model since it is more fitting than well-known distributions in educational attainment data , derived from a transformation on Monsef distribution. it is probability function has been generated also distribution function. reliability function and hazard function, some of statistical measures are investigated such like moments of origin point, central moments, Lorenz and Bonferroni curves also stress strengths parameter are investigated. parameter estimated is obtained by using more than method like maximum likelihood method, method of moment and least square method. the simulation scheme for unit Monsef parameters. Finally, real data is used for studying the flexibility of UMD also applied on UMD regression model compared with others well Known distributions.

1 Introduction

There are many phenomena have the uncertainty of a bounded, which lie on the interval (0, 1) such as the scores of ability tests and different rates. The transformation of the random variable generates new distribution with

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domain (0,1), which be more flexible and more fitting to data which lie in this interval [1,2]. Unit distributions are the best to describe this phenomenon one of the most important distributions that use in this data is the Beta distribution (or the Pearson type IV distribution) because of its flexibility there is many unit distributions which introduced as alternative to beta distribution and Kumaraswamy distribution like the unit-Gamma distribution [3], the unit-Logistic distribution [4], a gamma regression model [5] obtained depend on unit gamma distribution. unit Burr XII distribution and regression model [6] of the most important regression models are beta regression model [7] and Kumaraswamy regression model [8]. to in this paper a new unit distribution will be obtained depend on Monsef distribution [8], which has more flexibility, this paper has organized as follow, the first section introduced the method of transformation , pdf and cdf of the UMD, the survival and hazard rate and the behavior of the pdf and hazard function, the moments and ,Lorenz and Bonferroni curve are obtained in section 2, stress strength parameter were obtained in section 3, more than one estimation method were obtained to estimate the parameter of the new distribution in section 4, section 5 and 6 introduced the simulation study and application [9,10,11-13].

2 The Unit Monsef Distribution

Monsef distribution was introduced by Monsef [8] with cdf:

$$F(y) = 1 - \frac{e^{-x\theta}}{(\theta+1)^2+1}[(\theta(y+1)+1)^2+1], \qquad y, \theta > 0 \tag{1}$$

and the pdf of *Monsef* distribution is:

$$f(y) = \frac{\theta^3}{2+\theta(2+\theta)}(y+1)^2 e^{-y\theta}, \quad y, \theta > 0 \tag{2}$$

By using the transformation $X = \frac{y}{1+y}$ in equation (1) hence the new distribution was proposed with support on the unit interval. The cdf and the pdf of the UMD are:

$$F(x) = 1 - \frac{e^{\frac{x\theta}{-1+x}}\left(2+2x^2-2x(2+\theta)+\theta(2+\theta)\right)}{(-1+x)^2\left(2+\theta(2+\theta)\right)}, \quad 0 < x < 1, \ \theta > 0 \tag{3}$$

$$f(x) = \frac{e^{\frac{x\theta}{-1+x}}\theta^3}{(-1+x)^4(2+\theta(2+\theta))}, \qquad 0 < x < 1, \ \theta > 0 \tag{4}$$

The survival and hazard functions can be obtained as following:

$$S(x) = \frac{e^{\frac{x\theta}{-1+x}}(2+2x^2-2x(2+\theta)+\theta(2+\theta))}{(-1+x)^2(2+\theta(2+\theta))}$$

$$H(x) = \frac{\theta^3}{(-1+x)^2(2+2x^2-2x(2+\theta)+\theta(2+\theta))}$$

The behavior of the pdf and the hazard rate of the new distribution at different values of θ will be showing in the below figure.

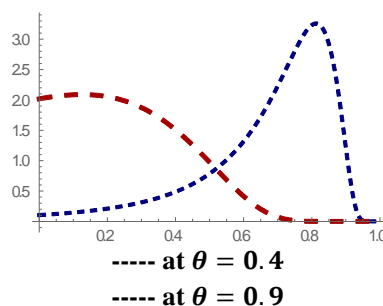
----- at $\theta = 0.4$
----- at $\theta = 0.9$

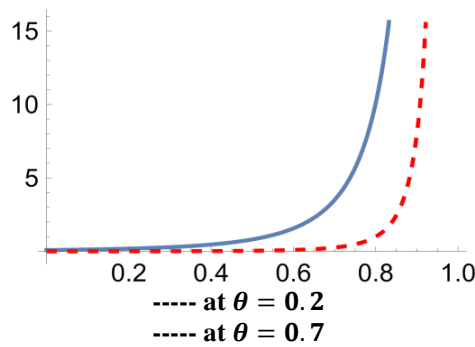**Fig. 1. The behavior of the unit Monsef pdf**

Fig. 2. The behavior of the unit Monsef hazard function

3 Statistical Measures

Some of the important statistical properties will be obtained in this section like moments, skewness, and Kurtosis coefficients, Lorenz and Bonferroni curves also the stress strength parameter were obtained.

3.1 Moments about the origin (raw moments)

The rth moment about the origin of the unit Monsef distribution is defined by

$$\acute{\mu}_r = \int_0^1 x^r \frac{e^{\frac{x\theta}{-1+x}}\theta^3}{(-1+x)^4(2+\theta(2+\theta))} \, dx, \quad r = 1,2,3,\dots$$

The first four moments about the origin are written as:

$$\acute{\mu}_1 = \frac{2+\theta}{2+\theta(2+\theta)}, \quad \acute{\mu}_2 = \frac{2}{2+\theta(2+\theta)}$$

$$\acute{\mu}_3 = \frac{2 + (-1+\theta)\theta + e^\theta\theta^3(\text{CoshIntegral}[\theta] - \text{SinhIntegral}[\theta])}{2+\theta(2+\theta)}$$

$$\acute{\mu}_4 = \frac{2 + \theta(-2+\theta(3+\theta)) + e^\theta\theta^3(4+\theta)(\text{CoshIntegral}[\theta] - \text{SinhIntegral}[\theta])}{2+\theta(2+\theta)}$$

3.2 Central moments

The rth moment about the mean of the unit Monsef distribution is defined by

$$\mu_r = \int_0^1 (x-\mu)^r \frac{e^{\frac{x\theta}{-1+x}}\theta^3}{(-1+x)^4(2+\theta(2+\theta))} \, dx$$

Where

the variance $(\sigma^2) = \mu_2 = \frac{\theta^2}{(2+\theta(2+\theta))^2}$

$$\mu_4 = \frac{\theta^3(8+\theta(3+\theta(3+\theta))(11+\theta(3+\theta)^2)+e^\theta\theta(2+\theta(2+\theta))^2(6+\theta(6+\theta))(\text{CoshIntegral}[\theta]-\text{SinhIntegral}[\theta]))}{(2+\theta(2+\theta))^4}$$

The skewness and kurtosis measures can be obtained by:

$$\beta_1 = (2+\theta(2+\theta))^4(\frac{(1+\theta)(4+\theta(2+\theta))}{(2+\theta(2+\theta))^2} + e^\theta(\text{CoshIntegral}[\theta] - \text{SinhIntegral}[\theta]))^2$$

$$\beta_2 = \frac{8}{\theta} + (3 + \theta(3 + \theta))(11 + \theta(3 + \theta)^2) + e^{\theta}(2 + \theta(2 + \theta))^2(6 + \theta(6 + \theta))(\text{CoshIntegral}[\theta] - \text{SinhIntegral}[\theta])$$

coefficient The of variation (CV) for the new distribution can be written as :

$$CV = \frac{\sqrt{\frac{\theta^2}{(2+\theta(2+\theta))^2}(2+\theta(2+\theta))}}{2+\theta}$$

3.3 Lorenz and Bonferroni curves

The Lorenz and Bonferroni curves have importance in studying economics and in reliability applications. The Lorenz curve for the unit Monsef distribution is defined as

$$L\big(F(x)\big) = \frac{2 + \frac{e^{\frac{x\theta}{-1+x}}(-2+x(-2+x-\theta)(-2+\theta)-\theta)}{(-1+x)^2} + \theta}{2+\theta}$$

The Bonferroni curve $B_F[F(x)]$ for the unit Monsef distribution is defined as

$$B_F[F(x)] = \frac{2 + \frac{e^{\frac{x\theta}{-1+x}}(-2+x(-2+x-\theta)(-2+\theta)-\theta)}{(-1+x)^2} + \theta}{(2+\theta)(1 - \frac{e^{\frac{x\theta}{-1+x}}(2+2x^2-2x(2+\theta)+\theta(2+\theta))}{(-1+x)^2(2+\theta(2+\theta))})}$$

3.4 Stress strength reliability

Studying the survival and failure rate under conditions is an important to the system the system will perform its intended function adequately. Let X and Y are two independent unit Monsef r.vs with parameters θ_1 and θ_2 then the stress strength measures can be obtained as:

$$R = (\theta_2^3(\theta_1^6 + \theta_1^5(6 + 4\theta_2) + \theta_1^4(20 + 20\theta_2 + 6\theta_2^2) + 2\theta_2^2(2 + \theta_2(2 + \theta_2)) + 2\theta_1\theta_2(5 + \theta_2)(2 + \theta_2(2 + \theta_2)) + \theta_1^2(2 + \theta_2(2 + \theta_2))(20 + \theta_2(10 + \theta_2)) + \theta_1^3(40 + 2\theta_2(25 + 2\theta_2(6 + \theta_2)))))/((2 + \theta_1(2 + \theta_1))(\theta_1 + \theta_2)^5(2 + \theta_2(2 + \theta_2)))$$

4 The Estimation Methods

In this section some methods of parameter estimation will be introduced for the new distribution unit Monsef.

4.1 Maximum likelihood estimation

Suppose there is a sample $x_1. \dots \dots \dots x_n$ of n iid observations. Coming from a distribution with probability density function identified in equation (4). The log–likelihood function is given by

$$ln\,\ell(x;\theta) = n\,ln\left[\frac{\theta^3}{(2+\theta(2+\theta))}\right] - \theta\sum_{i=1}^{n}\frac{x_i}{1-x_i} - 4\sum_{i=1}^{n}ln(1-x)$$

The partial derivatives of the log-likelihood for the parameter θ can be written:

$$\frac{\partial ln\ell(x;\theta)}{\partial\theta} = \frac{n(6+\theta(4+\theta))}{\theta(2+\theta(2+\theta))} - \sum_{i=1}^{n}\frac{x_i}{1-x_i}$$

The MLEs can be obtained by solving the derivatives of the $ln\,\ell(x;\theta)$ to zero. For the UMD the solution of the nonlinear equations has no closed form, also some numerical methods are needed for the solution.

4.2 The method of moment estimator

By letting $x_1, \ldots \ldots \ldots, x_n$ be a r.s with size n from the UMD, the moment estimates (MOM), for the parameter θ is given solving equation

$$\mu = \frac{2+\theta}{2+\theta(2+\theta)}$$

By solving the above equation

$$\hat{\theta} = \frac{(2+(2+\mu))(1-\frac{2(2+\mu)}{2+\mu(2+\mu)}+\sqrt{\frac{(2+\mu(4+\mu))^2}{(2+\mu(2+\mu))^2}})}{2(2+\mu)}$$

The solution of above equation has not closed form, but by using, μ we can compute the estimated values of θ.

4.3 Method of least square estimation

The LSE and WLSE were proposed by Swain, Venkatraman & Wilson [14] to estimate the parameters of Beta distributions.

The least square estimators can be obtained by minimizing

$$\sum_{j=1}^{n} \left[F(X_j) - \frac{j}{n+1} \right]^2$$

the least square estimator of θ can be introduced as:

$$\sum_{j=1}^{n} \left[1 - \frac{e^{\frac{\theta x_j}{-1+x_j}}(2+\theta(2+\theta)-2(2+\theta)x_j+2x_j^2)}{(2+\theta(2+\theta))(-1+x_j)^2} - \frac{j}{n+1} \right]^2$$

The partial derivatives of the Least square method for the parameters θ can be obtained as follow:

$$\frac{\partial \text{LSE}(x;\theta)}{\partial \theta} =$$
$$\sum_{j=1}^{n} 2 \left[1 - \frac{e^{\frac{\theta x_j}{-1+x_j}}(2+\theta(2+\theta)-2(2+\theta)x_j+2x_j^2)}{(2+\theta(2+\theta))(-1+x_j)^2} - \frac{j}{n+1} \right] \left(- \frac{e^{\theta(1+\frac{1}{-1+x_j})}\theta^2 x_j(6+\theta(4+\theta)+2x_j(-3-\theta+x_j))}{(2+\theta(2+\theta))^2(-1+x_j)^3} \right)$$

4.4 Method of Cramér- Von- Mises

Another method of estimation parameter is Cramér-von-Mises estimators of θ can be obtained by:

$$C(\theta, \lambda, p, c) = \frac{1}{12n} + \sum_{i=1}^{n} \left[F(x_{i:n}/\theta, \lambda, p, c) - \frac{2i-1}{2n} \right]^2$$

$$C(\theta, \lambda, p, c) = \frac{1}{12n} + \sum_{i=1}^{n} \left[1 - \frac{e^{\frac{\theta x_j}{-1+x_j}}(2+\theta(2+\theta)-2(2+\theta)x_j+2x_j^2)}{(2+\theta(2+\theta))(-1+x_j)^2} - \frac{2i-1}{2n} \right]^2$$

5 Simulation Study

A simulation Scheme for the UMD will be made by generating 5000 samples for the parameters θ at different sample sizes. The simulation nodes were chosen at different values at θ.

5.1 Simulation study for the new distribution using MLE

Table 1. The MSE and the Bias for the parameter θ

N	$\theta = 0.1$		$\theta = 0.75$		$\theta = 1.5$		$\theta = 3.5$	
	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias
10	0.00041	0.0010	0.0251	0.0319	0.1213	0.0775	0.9700	0.2518
20	0.00017	0.0038	0.0107	0.0084	0.0495	0.0210	0.3713	0.0826
30	0.00011	0.0009	0.0069	0.0079	0.0318	0.0099	0.2337	0.0761
40	0.00009	0.0006	0.0059	0.0053	0.0273	0.0269	0.2023	0.0750
50	0.00006	0.0004	0.0041	0.0034	0.0187	0.0192	0.1363	0.0609
60	0.00006	0.0002	0.0034	0.0032	0.0157	0.0127	0.1143	0.0569
70	0.00005	0.0002	0.0033	0.0029	0.0151	0.0077	0.1099	0.0399
80	0.00004	0.0003	0.0025	0.0022	0.0116	0.0068	0.0845	0.0255
90	0.00003	0.0004	0.0021	0.0021	0.0098	0.0059	0.0702	0.0241
100	0.00003	0.0003	0.0019	0.0011	0.0089	0.0058	0.0643	0.0214

5.2 Study for the new distribution using LSE method

Table 2. The MSE and the Bias for the parameter θ

N	$\theta = 0.1$		$\theta = 0.75$		$\theta = 1.5$		$\theta = 3.5$	
	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias
10	0.0042	0.0013	0.02599	0.01116	0.12817	0.02981	1.0691	0.12038
20	0.00021	0.00081	0.01247	0.00650	0.05923	0.01677	0.46404	0.06643
30	0.00014	0.00075	0.00827	0.00595	0.03902	0.01419	0.30041	0.05410
40	0.00011	0.00065	0.00664	0.00514	0.03119	0.01202	0.23976	0.04709
50	0.000085	0.00051	0.00511	0.00395	0.02384	0.00894	0.18094	0.03659
60	0.000069	0.00034	0.00416	0.00263	0.01935	0.00575	0.14495	0.02633
70	0.000059	0.00028	0.00354	0.00211	0.01647	0.00445	0.12309	0.02239
80	0.000053	0.00027	0.00317	0.00306	0.01466	0.00636	0.10929	0.02134
90	0.000046	0.00024	0.00274	0.00199	0.01267	0.00387	0.09454	0.02037
100	0.000042	0.00022	0.00248	0.00164	0.01146	0.00299	0.08563	0.01779

5.3 Simulation study for the new distribution using Cramér- Von- Mises method

Table 3. The MSE and the Bias for the parameter θ

N	$\theta = 0.1$		$\theta = 0.75$		$\theta = 1.5$		$\theta = 3.5$	
	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias
10	0.00042	0.0018	0.0261	0.0155	0.1296	0.0406	1.091	0.1570
20	0.00021	0.0011	0.0125	0.0087	0.0596	0.0224	0.4699	0.0854
30	0.00013	0.0009	0.0083	0.0075	0.0392	0.0180	0.3035	0.0671
40	0.00011	0.0008	0.0066	0.0063	0.0313	0.0149	0.2414	0.0568
50	0.00009	0.0006	0.0051	0.0048	0.0239	0.0112	0.1821	0.0444
60	0.00007	0.0004	0.0042	0.0034	0.0194	0.0076	0.1458	0.0329
70	0.00006	0.0003	0.0035	0.0027	0.0165	0.0061	0.1237	0.0281
80	0.00005	0.0004	0.0032	0.0036	0.0147	0.0078	0.1098	0.0323
90	0.00004	0.0003	0.0027	0.0025	0.0127	0.0052	0.0949	0.0248
100	0.00004	0.0002	0.0024	0.0021	0.0114	0.0041	0.0859	0.0218

The result of simulation shows that the MLE give the best results of biases and MSEs. LSE is the second good estimator, followed by the Cramér- Von- Mises method. we can argue that the MLE and least squares estimators are among the best performing estimators for UMD.

6 Regression Model and Application

In this section, a regression structure for the unit-Monsef distribution will be obtained, also a real data is applied to shows the flexibility of UMD. The MLE is used to estimate the parameters, The statistics criteria (WT, AIC, BIC, AICC, HQIC, AD and CVM), K-S and p-value (K-S) were calculated for the fitted distributions.

In regression analysis it is very common to model the mean of the response. Since the unit-Monsef distribution has closed form expression for mean it can be used in this context. It is noteworthy that the re-parametrized pdf of unit Monsef can be written as follow:

$$f(x,\mu) = \frac{e^{\frac{x(1-2\mu+\sqrt{1-4(-1+\mu)\mu})}{2(-1+x)\mu}}(1-6\mu+4\mu^2+\sqrt{1-4(-1+\mu)\mu})}{2(-1+x)^4\mu} \quad (5)$$

Let $Y_1,\ldots,Y_n$ be n independent random variables, where $Yi \sim UM(\mu_i), i = 1,\ldots,n$. The regression model is defined assuming that the mean of Yi satisfies the following functional relation

$$g(\mu_i) = x_i^\tau \beta$$

where $\beta = (\beta_1,\ldots,\beta_p)$ is a p-dimensional vector of regression coefficients $(p < n)$ and $x_i = (x_{i1},\ldots,x_{ip})$ denotes the observations on p known covariates.

6.1 The proposed regression model is

$$logit(\mu_i) = \delta_0 + \delta_1 x_{i1} + \delta_2 x_{i2} + \delta_3 x_{i3}, \quad i = 1,\ldots,30$$

6.2 The Education attainment

The used data set comes from the BLI of OECD countries. The data source is https://stats.oecd.org/index.aspx?DataSetCode=BLI. The educational attainment values of the OECD countries.

Table 4. The MLEs for the parameters, -log and K-S for different distributions

Distribution	MLEs	-log	K-S	P-value
Unit Monsef	$\hat{\theta} = 0.477$	-24.298	0.1013	0.9181
Unit Lindley	$\hat{\theta} = 0.3269$	-23.623	0.1559	0.4595
Kumaraswamy	$\hat{\alpha} = 6.154$	-21.096	0.7532	0.000
	$\hat{\beta} = 2.466$			
Beta	$\hat{\alpha} = 6.899$	-20.519	0.2027	0.1698
	$\hat{\beta} = 2.037$			
Unit Weibull	$\hat{\alpha} = 4.991$	-19.142	0.2101	0.1416
	$\hat{\beta} = 1.352$			

Table 5. The statistics criteria for different distributions

Distribution	AIC	BIC	AICC	HQIC	AD	CVM	WT
Unit Monsef	-46.596	-45.195	-45.596	-46.158	0.2537	0.03849	0.0384
Unit Lindley	-45.247	-43.846	-45.104	-44.799	0.42934	0.0724	0.05647
Beta	-37.519	-34.237	-36.595	-36.143	1.0531	0.1755	0.1383
Kumaraswamy	-38.192	-35.389	-37.747	37.295	67.523	6.895	1.443
Unit Weibull	-34.283	-33.839	-33.834	-33.387	1.2077	0.1975	0.1800

Unit Monsef Distribution

Unit Lindley Distribution

Beta Distribution

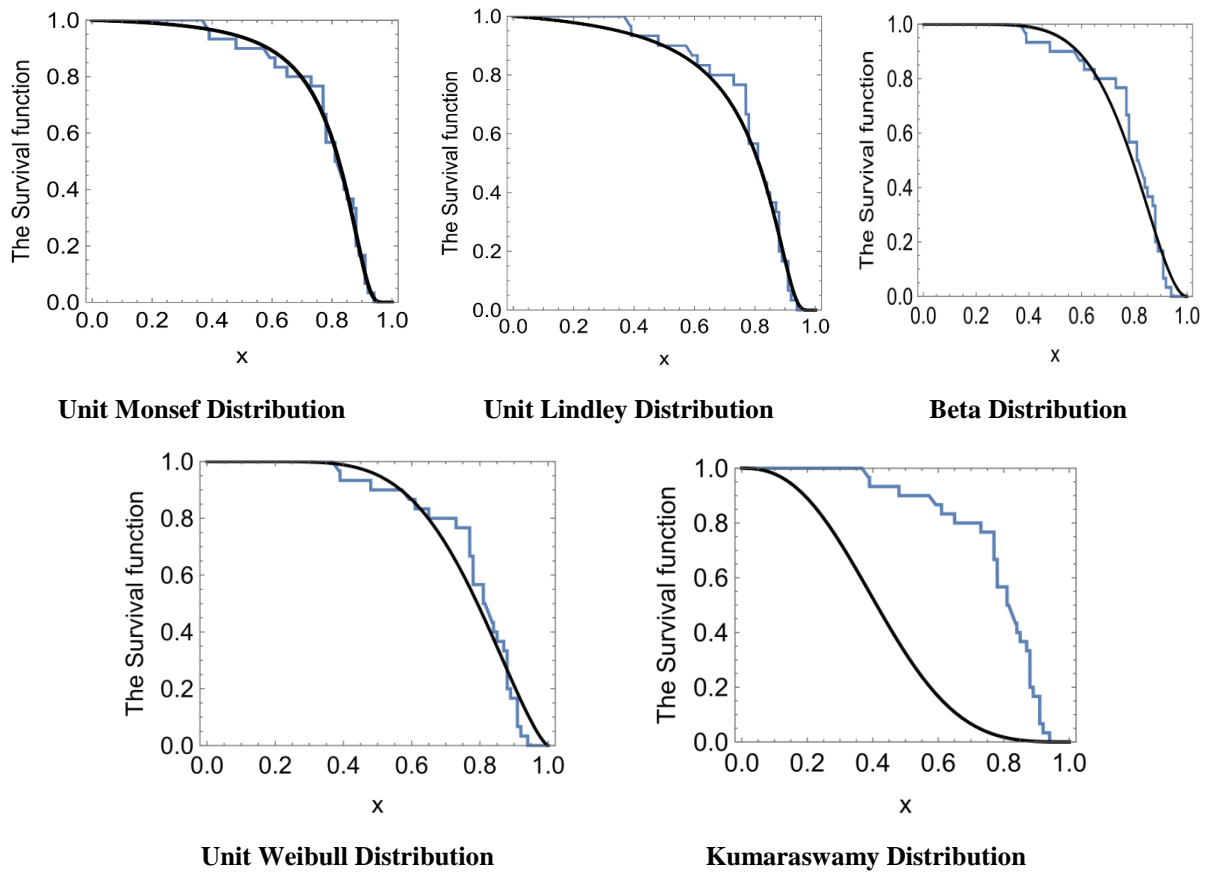Unit Weibull Distribution

Kumaraswamy Distribution

Fig 3. The survival curves for the educational attainment data

6.3 The Education attainment

The used data set comes from the BLI of OECD countries. The educational attainment values of the OECD countries (y) is considered as response (dependent) variable The goal is to explore the effects of following covariates on the conditional mean of the response variable: homicide rate (HR), dwellings without basic facilities (DWBF), and labor market insecurity (LMI). The logit link function which ensures that the estimated mean lies between 0 and 1, is used for all fitted regression models. The fitted regression model is

$$Logit(\mu_i) = \beta_0 + \beta_1\, HR + \beta_2\, DWBF + \beta_3 LMI$$

Table 6. The estimated value of the covariates for the Unit Monsef regression model comparing regression models

MLE	Unit Monsef	Unit Lindley	Unit Weibull	Kumaraswamy	Beta
β	-------	-------	1.8894	3.2183	13.1724
δ_0	1.668	1.502	1.9856	1.8935	1.7749
δ_1	0.0213	-0.0241	0.05202	0.00756	-0.0216
δ_2	-0.0320	-0.0288	-0.09362	-0.06828	-0.0618
δ_3	-0.0446	-0.0411	-0.0667	0.04769	-0.0451

Table 7. The estimated criteria for the Unit Monsef regression model comparing regression models

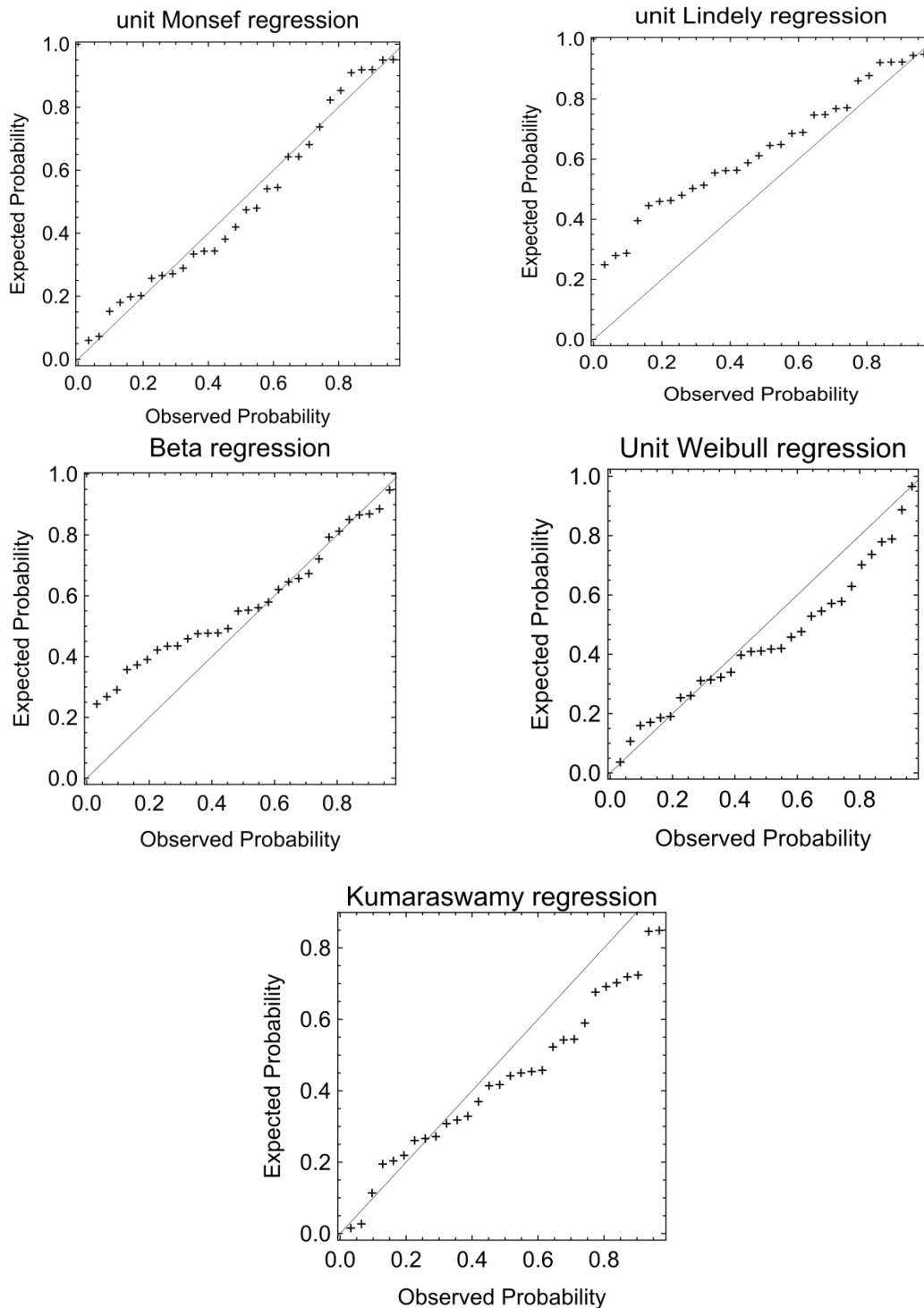| Criteria	Unit Monsef	Unit Lindley	Unit Weibull	Beta	Kumaraswamy
AIC	-47.1035	-44.045	-45.352	-42.2007	-39.4771
BIC	-41.4987	-38.440	-38.346	-35.1947	-32.4711
-log	-27.5518	-26.023	-27.676	-26.1004	-24.7385

Fig. 4. Q-Q Plots for the educational attainment data

The results of the above shows that the UMD is more flexible and more fitting comparing with other distributions, since has the smallest statistic criteria and the smallest log likelihood, hence the above values of measures will lead to the pdf of the UMD has its own shape and may be difficult to replace by any other known distribution.

7 Conclusion

In this paper, the pdf and cdf of UMD were derived in section 2 also the behavior of the pdf and hazard rate are drawn at different values of the parameter. The most important statistical properties such like moments, stress strength parameters and other are derived in section 3. The parameter estimates is obtained using maximum likely hood, method of moment, least square method, weighted least square and Cramér- Von- Mises method in section 4. In section 5 simulation study is introduced using MLE, LSE and Cramér- Von- Mises method, Finally the regression model for the proposed distribution is obtained and a real data is applied in UMD and compared with other well known distribution and the results show that the UMD is more flexible and well fit then the other distributions in section 6**.**

Competing Interests

Authors have declared that no competing interests exist.

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