Physical Science International Journal

23(3): 1-13, 2019; Article no.PSIJ.51873
ISSN: 2348-0130

Fermion Colour and Flavour Originating from Multiple Representations of the Lorentz Group and Clifford Algebra

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Author's contribution

The sole author designed, analyzed, interpreted and prepared the manuscript.

Article Information

DOI: 10.9734/PSIJ/2019/v23i330158

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Complete Peer review History: http://www.sdiarticle4.com/review-history/51873

Received: 08 August 2019
Accepted: 14 October 2019
Published: 19 October 2019

Original Research Article

ABSTRACT

Where do such fermion properties as colour and flavour come from? We attempt to give a possible answer to this question in our paper. For that purpose we use the reducible $(\frac{1}{2}, \frac{1}{2})$ representation of the Lorentz group. Then the fermion corresponds to a doublet, each component of which can be described by the standard Dirac equation. In this way we conclude that quark and lepton, when being considered as doublets, originate from the discussed multiple representations of the Lorentz group (LG) and the related Clifford algebra. In particular the threefold colour degree of freedom emerges naturally, and similarly the threefold generation degree, both being enabled essentially by the fact that the $SU(2)$ group has three generators given by the Pauli matrices. The Dirac spinor, or for zero mass the chiral Weyl spinor, remains the building block of that theory.

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Keywords: Quantum mechanics; relativistic wave equation; models beyond the standard model.

PACS Physics Subject Classification: 03.65-w; 03.65-Pm; 12.60.-i.

1 INTRODUCTION

Among the key open questions in modern elementary particle physics are those basic ones addressing the physical origin of such striking fermion properties as colour and flavour. Why do quarks just come in three colours, and why do leptons and quarks occur in three generations or six flavours? Why are all elementary fermions found empirically to be ordered in doublets, like the electron and neutrino, or the up and down quark, as well as their heavier relatives? What are the physical reasons for the appearance of the two gauge groups $SU(2)$ and $SU(3)$? These important questions have been around for decades, and are competently discussed for the non-expert physicist or educated layman in the excellent popular books by [1] and [2], and of course in depth addressed in modern textbooks of quantum field theory (QFT), like the ones by [3] and [4]. A lucid description of the history and detection of quark colour was recently given by [5], including the important relevant references.

We attempt to give a possible answer to the above questions in our paper. In brief, these properties are all connected and originate from permutation symmetries associated with the reducible $(\frac{1}{2}, \frac{1}{2})$ representations of the Lorentz group and related Clifford algebra. Yet, the Dirac spinor, or for zero mass the chiral Weyl spinor, remains the building block of the extended theory. The fundamental $SU(2)$ group and the representation of the angular momentum algebra in terms of the Pauli matrix vector [6], $\boldsymbol{\sigma} = (\sigma_{\mathrm{x}}, \sigma_{\mathrm{y}}, \sigma_{\mathrm{z}})$, plays an eminent role in this subject. Permutation of its three components naturally yields the empirically rather puzzling threefold multiplicity of colour and flavour. But to establish this notion requires to consider the fermion at the outset as a doublet.

Thus both colour and flavour essentially emerge from this duality, and owing to the simple fact that the related $SU(2)$ group has three generators. These traits are not included in the standard Dirac equation (in Weyl or Dirac representation), which describes the fermion as a doublet of particle and antiparticle, both having a spin either up or down. However, spin is not a relativistic property, but just related to the spinor representation of the rotation group $SO(3)$, which is obvious from the relation

$$(\boldsymbol{\sigma} \cdot \mathbf{p})^2 = \mathbf{p}^2 1_2, \tag{1.1}$$

with the momentum vector $\mathbf{p}$ and 2×2 unit matrix 1_2. The doublet nature of the fermion is intimately related to the structure of space-time, and naturally emerges in the symmetric $SU(2) \oplus SU(2)$ representation of the $SO(3,1)$ Lorentz group.

When considering the Clifford algebra subsequently, we do not have to pay attention to the fermion mass m, the square of which is a Casimir operator, i.e., an invariant property of a particle under Lorentz transformation (LT). But all fermions occurring in the standard model (SM) including the neutrino (as inferred from neutrino oscillations, see e.g. [7]) are known to have mass, though showing a huge spread in their values, which can now be calculated ab initio within the SM, for references see e.g. the paper by [8]. According to these calculations baryon masses can be understood as arising mainly from "condensation" of gauge-field energy, whereas lepton and quark masses are mainly determined by the Yukawa coupling to the Higgs field [4].

These important issues are not dealt with here, but we shall concentrate on the origin of the multiplicity of the Clifford algebra that is mirrored in the fermion properties. We start with two more tutorial sections on the generators of the LG and the Dirac equation in general abstract form. Then we discuss the various forms of the Clifford algebra, yielding different versions of the Dirac equation. Two appendices address specific issues and provide relevant matrices appearing in the representations of the LG.

2 THE FOUR-VECTOR GENERATORS OF THE LORENTZ GROUP

The purpose of this introduction is to remind the reader of the origin of the generators of the Lorentz group (LG) from the four-vector representation in Minkowski space. We therefore quote in the appendix for completeness the component matrices of the hermitian three-vector rotation operator $\mathbf{J}$, which is the generator of the $SO(3)$ rotation subgroup of the Lorentz group, and of the anti-hermitian three-vector boost operator $\mathbf{K}$. According to their definitions, the rotation and boost operators obey the well known linked three-vector equations of the Lorentz algebra, which can be written concisely as $\mathbf{J} \times \mathbf{J} = \mathrm{i}\mathbf{J},\ \ \mathbf{K}\times\mathbf{K} = -\mathrm{i}\mathbf{J},\ \ \mathbf{J}\times\mathbf{K} = \mathbf{K}\times\mathbf{J} = \mathrm{i}\mathbf{K}$. We can thus define the following linear combinations

$$\mathbf{J}_{\pm} = \frac{1}{2}(\mathbf{J} \pm \mathrm{i}\mathbf{K}), \tag{2.1}$$

which obey the corresponding relations

$$\mathbf{J}_{\pm} \times \mathbf{J}_{\pm} = \mathrm{i}\mathbf{J}_{\pm}, \qquad \mathbf{J}_{\pm} \times \mathbf{J}_{\mp} = 0. \tag{2.2}$$

These commutation relations are constitutive for the Lie algebra $so(3,1) = su(2) \oplus su(2)$ associated with the Lorentz Transformation (LT). Apparently, this Lie algebra can be decomposed into two commuting $su(2)$ sub-algebras consisting of the generators of the $SU(2)$ group.

The related 4×4-matrices $\mathbf{J}_{\pm}$ define generators of the irreducible $SU(2) \oplus SU(2)$ representation of the LG in Minkowski spacetime. These symmetric (and constitutive for the Lorentz algebra) four-vector generators can be rewritten after [17] as matrix operator $\mathbf{J}_{\pm} = \frac{1}{2}\mathbf{\Sigma}_{\pm}$, with $\mathbf{J}_{\pm}^2 = s(s+1)1_4$ and $s = \frac{1}{2}$. Here 1_4 means the 4×4 unit matrix. We shall call $\mathbf{J}_{+}$ the right-chiral, respectively, $\mathbf{J}_{-}$ the left-chiral spin operator, involving the novel and generalized 4×4 spin matrices,

$$\Sigma_{\pm\,\mathrm{x}} = \begin{pmatrix} 0 & \pm1 & 0 & 0 \\ \pm1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathrm{i} \\ 0 & 0 & \mathrm{i} & 0 \end{pmatrix},\ \Sigma_{\pm\,\mathrm{y}} = \begin{pmatrix} 0 & 0 & \pm1 & 0 \\ 0 & 0 & 0 & \mathrm{i} \\ \pm1 & 0 & 0 & 0 \\ 0 & -\mathrm{i} & 0 & 0 \end{pmatrix},\ \Sigma_{\pm\,\mathrm{z}} = \begin{pmatrix} 0 & 0 & 0 & \pm1 \\ 0 & 0 & -\mathrm{i} & 0 \\ 0 & \mathrm{i} & 0 & 0 \\ \pm1 & 0 & 0 & 0 \end{pmatrix}, \tag{2.3}$$

with the commutator $[\mathbf{\Sigma}_{\pm\,i}, \mathbf{\Sigma}_{\mp\,j}] = 0$. Also, $\mathbf{\Sigma}_{\pm} \times \mathbf{\Sigma}_{\pm} = 2\mathrm{i}\mathbf{\Sigma}_{\pm}$. By complex conjugation of the Sigma matrices in (2.3), we can see that they obey $(\mathbf{\Sigma}_{\pm})^* = -\mathbf{\Sigma}_{\mp}$. Moreover, the Sigma matrices fulfill, like the Pauli matrices, a metric condition in real space, namely

$$\Sigma_{\pm\,j}\Sigma_{\pm\,k} + \Sigma_{\pm\,k}\Sigma_{\pm\,j} = 2\delta_{j,k}1_4. \tag{2.4}$$

Thus the sigma component matrices squared give unity, and their sum yields, $\mathbf{\Sigma}_{\pm}^2 = 3\,1_4$. The related four-vector Lorentz transformation is a real 4×4 matrix operator since it operates on a real four-vector V^{μ} in Minkowski space. Finally note that the matrices in (2.3) cannot together be made block-diagonal, which is obvious from their origin in the matrices of $\mathbf{J}$ and $\mathbf{K}$ given in the appendix. This fact simply reflects the $so(3,1)$ algebra.

According to the seminal work by [9], [10], [11], and [12], one can extend the above representation of the LG by using any other adequate version of the involved $su(2)$ algebra, which is the fundamental one for angular momentum. Consequently, various more general representations of the Lorentz group can be constructed and then classified as $(\frac{m}{2}, \frac{n}{2})$ with integer m and n. This subject is dealt with exhaustively in the cited papers and in any modern textbook [4, 3] of quantum field theory (QFT).

In what follows, we will construct some novel versions of the Lorentz algebra and the related matrix representations of the Clifford algebra, among them some which involve the original four space-time dimensions of the Minkowski space. These matrices then act on complex four-component vectors. Conveniently, we shall call the related vector doublets Minkowski spinors. They are reducible to doublets of Dirac spinors.

3 REVISITING THE DIRAC EQUATION DESCRIBED IN TERMS OF THE ABSTRACT CLIFFORD ALGEBRA

This section does contain merely things that can be found in any modern textbook on QFT [4, 3], Yet we believe it is helpful in and needed for setting the scene for the subsequent key topics appearing in this paper. Here we are concerned with a discussion of the main properties of the Dirac equation, yet based solely on the abstract Clifford algebra. Various specific matrix representations of this algebra are considered subsequently. For a free fermion (i.e., a spin 1/2 particle) of mass m the key Casimir operator of the LG is the mass squared, which leads to the so called mass-shell condition of the four-momentum as follows

$$p^{\mu}p_{\mu} = m^2, \tag{3.1}$$

with $p_{\mu} = (E, -\mathbf{p})$. This relation can be linearized by help of the Clifford algebra which can be expressed in standard terms as

$$\Gamma^{\mu}\Gamma^{\nu} + \Gamma^{\nu}\Gamma^{\mu} = 2\,g^{\mu\nu}. \tag{3.2}$$

Then we can write

$$\Gamma^{\mu}p_{\mu} = m, \;\; \Gamma^{\mu} = (\Gamma_0, \mathbf{\Gamma}). \tag{3.3}$$

Here $g^{\mu\nu}$ is the metric tensor in Minkowski space in standard notation. When we now square Eq. (3.3) and use the metric properties of the above Clifford algebra, we retain the Casimir operator (3.1). The Dirac [13] equation is obtained from (3.3) by insertion of the the relativistic quantum mechanical four-momentum operator as $P_{\mu} = \mathrm{i}\partial_{\mu} = \mathrm{i}(\frac{\partial}{\partial t}, \frac{\partial}{\partial \mathbf{x}})$, whereby we use standard symbols, notations and conventional units as in the textbooks [4, 3] for quantum field theory, and we also set $\hbar = c = 1$. Thus we obtain for the fermion the linear covariant wave equation

$$\Gamma^{\mu}P_{\mu}\Psi = \Gamma^{\mu}\mathrm{i}\partial_{\mu}\Psi = m\Psi, \tag{3.4}$$

for the spinor Ψ. Conventionally, one introduces the so called Γ_5 matrix as $\Gamma_5 = \mathrm{i}\Gamma_0\Gamma_{\mathrm{x}}\Gamma_{\mathrm{y}}\Gamma_{\mathrm{z}}$, which obeys $(\Gamma_5)^2 = 1$ and mutually anticommutes with the four other Gamma matrices by definition. We may by its help also define the important projection operator

$$P_{\pm} = \frac{1}{2}(1 \pm \Gamma_5), \tag{3.5}$$

which is idempotent and has the effect that $P_{\pm}\Gamma^{\mu} = \Gamma^{\mu}P_{\mp}$, i.e., its sign switches by commutation with the Gamma matrices. Conventionally, Γ_5 bears the name chirality operator. The reason being that according to (5.17) in the appendix, the spinor Ψ can be decomposed into its right-chiral part $\Psi_{\mathrm{R}} = P_{-}\Psi$ and left-chiral part $\Psi_{\mathrm{L}} = P_{+}\Psi$, which always transform independently under the LT. Related relevant information about the spinorial Lorentz transformation of the Dirac equation in abstract form is contained in the appendix.

4 VARIOUS VERSIONS OF THE CLIFFORD ALGEBRA FOR A FERMION

4.1 The Standard Dirac Equation for a Fermion and Various Clifford Algebras

At this point we have to remind the reader that in the Dirac equation [13, 4] the two simplest possible spinor representations of the LG are employed. They are given by the two generator pairs for the rotation operator $\mathbf{J} = \frac{1}{2}\boldsymbol{\sigma}$ and the boost operator $\mathbf{K} = \pm\frac{\mathrm{i}}{2}\boldsymbol{\sigma}$. They are based on the fundamental two-dimensional representation of $SU(2)$ as generated by the Pauli matrix vector [6], which acts on left- and right-chiral two-component Weyl spinors usually denoted as $\phi_{\mathrm{R,L}}$. It is assumed that either $\mathbf{J}_{+} = \mathbf{J}$, and the trivial one then is $\mathbf{J}_{-} = 0$ or vice versa, which just yield the two well known asymmetric $(\frac{1}{2}, 0)$ or $(0, \frac{1}{2})$ irreducible representations of the LG. Their matrix dimensions are reduced to two instead of four as in the original four-vector representation given above in (2.3). However, the use of the combined fundamental and trivial representation of $SU(2)$ breaks at the outset chiral symmetry, which is yet guaranteed if the $\mathbf{J}_{\pm}$ are treated equally, like in the genuine representation (2.3).

In this paper we shall make frequent use of the three hermitian Pauli matrices again (defining the $SU(2)$ group generators), but we will give them, to avoid confusion with the nomenclature, the new name tau matrices defined as follows

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \ \tau_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \ \ \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4.1}$$

Of course we have $\tau_i\tau_j + \tau_j\tau_i = 2\delta_{i,j}1_2$, and thus $\tau_i^2 = 1_2$. Moreover, $\tau_1\tau_2 = \mathrm{i}\tau_3$, cyclically. Furthermore, we introduce the associated similarity transformations

$$U_j = \frac{1}{\sqrt{2}}(1_2 - \mathrm{i}\tau_j), \ \ U_j^\dagger = U_j^{-1} = \frac{1}{\sqrt{2}}(1_2 + \mathrm{i}\tau_j), \tag{4.2}$$

with $j = 1, 2, 3$, and with $U_jU_j^{-1} = 1_2$. Thus we obtain the cyclic relations

$$U_1\tau_2U_1^{-1} = \tau_3, \ \ U_2\tau_3U_2^{-1} = \tau_1, \ \ U_3\tau_1U_3^{-1} = \tau_2, \tag{4.3}$$

for the tau matrices. And similarly, we find that

$$U_1\tau_3U_1^{-1} = -\tau_2, \ \ U_2\tau_1U_2^{-1} = -\tau_3, \ \ U_3\tau_2U_3^{-1} = -\tau_1. \tag{4.4}$$

We are going to use these expressions throughout the paper. Given the Pauli matrices for the physical spin vector $\boldsymbol{\sigma}$, we can easily write down exactly three different gamma-matrix doublets, with $\gamma^\mu = (\gamma_0, \boldsymbol{\gamma})$, as follows

$$\begin{array}{l} \gamma_{01} = \tau_1 \otimes 1_2, \ \ \boldsymbol{\gamma}_2 = \mathrm{i}\tau_2 \otimes \boldsymbol{\sigma}, \\ \gamma_{01} = \tau_1 \otimes 1_2, \ \ \boldsymbol{\gamma}_3 = \mathrm{i}\tau_3 \otimes \boldsymbol{\sigma}, \\ \gamma_{02} = \tau_2 \otimes 1_2, \ \ \boldsymbol{\gamma}_3 = \mathrm{i}\tau_3 \otimes \boldsymbol{\sigma}, \\ \gamma_{02} = \tau_2 \otimes 1_2, \ \ \boldsymbol{\gamma}_1 = \mathrm{i}\tau_1 \otimes \boldsymbol{\sigma}, \\ \gamma_{03} = \tau_3 \otimes 1_2, \ \ \boldsymbol{\gamma}_1 = \mathrm{i}\tau_1 \otimes \boldsymbol{\sigma}, \\ \gamma_{03} = \tau_3 \otimes 1_2, \ \ \boldsymbol{\gamma}_2 = \mathrm{i}\tau_2 \otimes \boldsymbol{\sigma}. \end{array} \tag{4.5}$$

The first is known as the Weyl representation, the sixth as the Dirac representation. The other four bear no name yet and have to our knowledge not been used in the literature. In the sequence given above, they are obtained by cyclic permutation of the index pairs at the gamma matrices with a spatial index. The threefold multiplicity just reflects the fact that the $SU(2)$ group has exactly three generators. We suggest that this striking permutation symmetry corresponds to the "flavour" degrees of freedom (which are just twice three) of a fermion in the standard model. By means of the similarity transformations in (4.3) and (4.4), the representations are all mutually connected, separate from unimportant phase factors of plus or minus.

To give only one well known example, the corresponding equations for the standard Dirac spinor ($\psi^\dagger = (\phi_1^\dagger, \phi_2^\dagger)$) read in the Weyl, also named chiral, representation in terms of Pauli spinors as follows

$$\begin{array}{l} \mathrm{i}(\frac{\partial}{\partial t} - \boldsymbol{\sigma} \cdot \frac{\partial}{\partial \mathbf{x}})\phi_1 = m\phi_2 \\ \mathrm{i}(\frac{\partial}{\partial t} + \boldsymbol{\sigma} \cdot \frac{\partial}{\partial \mathbf{x}})\phi_2 = m\phi_1 \end{array} . \tag{4.6}$$

For vanishing mass, $m = 0$, the equations decouple. Permutations among the six representations in (4.5) would not change the physical content of the Dirac equation, describing a particle and its antiparticle with their spin up and down duality.

The dual nature of the representations in (4.5) suggests to lump them together into three doublets, because the difference in representation may not be a mathematical redundancy but have physical meaning. This idea was already proposed by [14]. This notion becomes even more convincing by the results of the subsequent sections. Therefore, we may write an extended Dirac equation for the fermion as a doublet in the form

$$\Gamma_1^\mu \mathrm{i}\partial_\mu \Psi_1 = m\Psi_1, \ \ \Gamma_1^\mu = \begin{pmatrix} (\gamma_{01}, \boldsymbol{\gamma}_2) & 0 \\ 0 & (\gamma_{01}, \boldsymbol{\gamma}_3) \end{pmatrix}, \tag{4.7}$$

with $\Psi_1^\dagger = (\psi_2^\dagger, \psi_3^\dagger)$. Now any of the three couples which can be composed of the six gamma matrices in (4.5) may be used. Cyclic permutation of the indices in (4.7) yields three different versions, which we interpret as corresponding to three fermion generations. By help of (4.3) and (4.4), equation (4.7) can also be written as $\gamma^\mu \mathrm{i}\partial_\mu \tilde{\Psi} = m\tilde{\Psi}$, with $\gamma^\mu = (\gamma_{01}, \boldsymbol{\gamma}_3)$, and $\tilde{\Psi}^\dagger = ((U_1 \otimes 1_2\, \psi_2)^\dagger, \psi_3^\dagger)$. Therefore, the doublet theory is related to the $SU(2)$ symmetry, to which the Yang-Mills [15] gauge theory can be applied. This result is in compliance with the Coleman-Mandula theorem [16], after which the connected symmetry of a field can only be a direct product of the internal symmetry group with the Lorentz (Poincaré) group.

4.2 Different Gamma Matrices for the Lepton

The aim of this section is to construct various versions of the Clifford algebra for a fermion based on the chiral spin matrices (2.3). In order to do so we have to use both spin matrices and employ them on an equal footing in the subsequent calculation. Our goal can be achieved by a linear combination of them, while acting both on a complex Minkowski vector. So we define the new hermitian 8×8-matrix vector operator

$$\boldsymbol{\Sigma}_1 = \frac{1}{\sqrt{2}}(\tau_2 \otimes \boldsymbol{\Sigma}_+ + \tau_3 \otimes \boldsymbol{\Sigma}_-). \qquad (4.8)$$

The factor $\sqrt{2}$ is required to normalize Sigma such that we have $(\boldsymbol{\Sigma}_1)^2 = 1_4$. We stress that $\boldsymbol{\Sigma}_1$ has the same metric properties as the Pauli matrices, yielding

$$\Sigma_{1\,j}\Sigma_{1\,k} + \Sigma_{1\,k}\Sigma_{1\,j} = 2\delta_{j,k}1_8, \qquad (4.9)$$

whereby the fact that the commutator $[\Sigma_{\pm\,j}, \Sigma_{\mp\,k}]$ vanishes has been exploited. But note that these sigma matrices do not obey the $su(2)$ spin algebra, and thus $\frac{1}{2}\boldsymbol{\Sigma}_1$ is not a spin operator. It just corresponds to a linear combination of the right-chiral and left-chiral spin operators $\mathbf{J}_+$ and $\mathbf{J}_-$. Yet in order to ensure the essential property (4.9), the coefficients τ_2 and τ_3 in (4.8) ought to be Pauli matrices, because the scalar product of the chiral spin matrices is a non-vanishing diagonal matrix $\boldsymbol{\Sigma}_+ \cdot \boldsymbol{\Sigma}_- = \mathrm{diag}[-3, 1, 1, 1]$. We are now in the position to define the desired Gamma matrices for the Dirac equation by tensor multiplication in the following way

$$\Gamma_0 = \tau_1 \otimes 1_4, \quad \Gamma_0^\dagger = \Gamma_0, \quad \Gamma_0^2 = 1_2 \otimes 1_4 = 1_8 \qquad (4.10)$$

and similarly

$$\boldsymbol{\Gamma} = \mathrm{i}\boldsymbol{\Sigma}_1, \quad \boldsymbol{\Gamma}^\dagger = -\boldsymbol{\Gamma}, \quad \boldsymbol{\Gamma}^2 = -3\,1_2 \otimes 1_4 = -3\,1_8, \qquad (4.11)$$

in formal analogy to the usual Dirac γ matrices. By definition we have $\Gamma^\mu\Gamma_\mu = (\Gamma^0)^2 - \boldsymbol{\Gamma}^2 = 4\,1_8$. Given that the chiral spin discussed above has four components, we required two more degrees of freedom to construct these Gamma matrices. They correspond of course to the particle and antiparticle doublet, as we know it well from the standard Dirac equation. The Dirac equation based on the above Clifford algebra has been studied extensively by [17].

Close inspection of equations (4.8), (4.10) and (4.11) reveals a striking permutation symmetry, namely we can permute the indices of the tau matrices without changing the physics. By definition, the tau matrices are connected by the formula $\mathrm{i}\tau_1 = \tau_2\tau_3$, whereby the indices can be cyclically permuted.

Application of the transformations (4.2), (4.3) and (4.4) on the Gamma matrix (4.10) and (4.11) yields three physically equivalent representations of the Clifford algebra. As there are only three $SU(2)$ generators, we obtain consequently a triple of Gamma matrices. The lepton is apparently coming in six "flavours" or three generations. This result is in agreement with the key empirical property of elementary particle physics that the leptons in the SM come in six flavours. The multiplicity originates from, and in fact is enabled by, the chiral spin (2.3), i.e., by the notion that the lepton exists as a doublet.

Use of the chiral spin matrices (2.3) leads to somewhat awkward algebra, and therefore it seems more convenient to make use of the Pauli matrices to describe the physical spin. For example, when using $\Gamma_1^\mu = (\tau_1 \otimes 1_4, \mathrm{i}\boldsymbol{\Sigma}_1)$, we obtain the connected equations (with the Minkowski spinor $\Phi^\dagger = (\Phi_1^\dagger, \Phi_2^\dagger)$) for the particle and antiparticle in the form

$$\begin{array}{l} \mathrm{i}(\frac{\partial}{\partial t} - \frac{1}{\sqrt{2}}\boldsymbol{\Sigma}_+ \cdot \frac{\partial}{\partial \mathbf{x}})\Phi_1 = (-\frac{1}{\sqrt{2}}\boldsymbol{\Sigma}_- \cdot \frac{\partial}{\partial \mathbf{x}} + m)\Phi_2 \\ \mathrm{i}(\frac{\partial}{\partial t} + \frac{1}{\sqrt{2}}\boldsymbol{\Sigma}_+ \cdot \frac{\partial}{\partial \mathbf{x}})\Phi_2 = (+\frac{1}{\sqrt{2}}\boldsymbol{\Sigma}_- \cdot \frac{\partial}{\partial \mathbf{x}} + m)\Phi_1 \end{array} . \quad (4.12)$$

These twin equations are analogous to the standard Weyl ones given in (4.6). When multiplying them up and exploiting the properties of the Sigma matrices, we of course retain the Klein-Gordon [18] equation for each of them. The above chirally symmetric equations can be made more transparent by an adequate replacement of the chiral spin by $\boldsymbol{\sigma}$. When solving them in terms of the plane wave solutions, we have to deal with the helicities $H_\pm = \boldsymbol{\Sigma}_\pm \cdot \hat{\mathbf{p}}$, with the momentum unit vector $\hat{\mathbf{p}}$. Obviously, $H^2 = 1_4$, and thus the eigenvalues are ± 1 and twofold degenerate. As the two chiral spin matrices in (2.3) commute, the $H_\pm$ can have common eigenfunctions. The following four eigenvalue pairs are thus possible in (4.8) and (4.12), namely

$$(+,+), (+,-), (-,+), (-,-).$$

They can be captured and described by the replacement

$$\tau_2 \otimes \boldsymbol{\Sigma}_+ + \tau_3 \otimes \boldsymbol{\Sigma}_- \longmapsto (\tau_2 \pm \tau_3) \otimes 1_2 \otimes \boldsymbol{\sigma}. \quad (4.13)$$

This replacement leaves the physical content unchanged. If we introduce $\tau_\pm = \frac{1}{\sqrt{2}}(\tau_2 \pm \tau_3)$, with $\tau_\pm^2 = 1_2$ and $\tau_1\tau_\pm + \tau_\pm\tau_1 = 0$, and $\tau_+\tau_- = -\mathrm{i}\tau_1$, we can decompose the Minkowski spinor Φ into a doublet of two Dirac spinors, $\Phi^\dagger = (\psi_+^\dagger, \psi_-^\dagger)$. They obey the Dirac equation coming in a new representation guise in a right- and left-chiral version as

$$(\tau_1(\mathrm{i}\frac{\partial}{\partial t}) + \mathrm{i}\tau_\pm(\boldsymbol{\sigma} \cdot \mathrm{i}\frac{\partial}{\partial \mathbf{x}}))\psi_\pm = m\psi_\pm. \quad (4.14)$$

Since $U_1\tau_\pm U_1^{-1} = -\tau_\mp$, these versions are physically equivalent, including a spin flip, and thus the chiral invariance of (4.14) becomes obvious. Furthermore, a permutation of the indices at the taus indicates that three representations are possible, corresponding to the already mentioned flavour degrees. Yet, it is not obvious to us whether they can be connected by a similarity transformation.

The main conclusion of this analysis is that, when the chiral symmetry is enforced at the outset and maintained, the fermion comes as a doublet, and thus has two new independent degrees of freedom, in addition to the common four degrees described by a single Dirac spinor.

4.3 Different Gamma Matrices for the Quarks

Are there possible four-dimensional representations of the LG generators other than the genuine ones, which are given by the normal LT in Minkowski space-time after (2.2) and (2.3)? To recall, we are looking for mathematical objects obeying the angular momentum commutation relation (2.2) and commute with each other. Indeed one can obtain lucid representations, where the requested 4×4 matrix is constructed by tensor multiplication. Let us first define the following projection operators

$$P_j^\pm = \frac{1}{2}(1_2 \pm \tau_j); \quad (P_j^\pm)^2 = P_j^\pm. \quad (4.15)$$

The index j runs from 1 to 3. The idempotence is the key enabling property, since we can define

$$\mathbf{J}_{\pm\, j} = \frac{1}{2}P_j^\pm \otimes \boldsymbol{\sigma}. \quad (4.16)$$

Then we obtain by taking the cross product of this three-vector the result

$$\mathbf{J}_{\pm\, j} \times \mathbf{J}_{\pm\, j} = \frac{1}{4}(P_j^\pm)^2 \otimes (\boldsymbol{\sigma} \times \boldsymbol{\sigma}) = \mathrm{i}\frac{1}{2}P_j^\pm \otimes \boldsymbol{\sigma} = \mathrm{i}\mathbf{J}_{\pm\, j}. \quad (4.17)$$

Since $P_j^\pm P_j^\mp = 0$, we herewith ensure that $\mathbf{J}_{\pm\, j} \times \mathbf{J}_{\mp\, j} = 0$. These three representations of the Lorentz algebra are connected through similarity transformations which were already presented in equations (4.2) and (4.3). They can easily be transferred to the above projection operators. This procedure yields

$$U_1 P_2^{\pm} U_1^{-1} = P_3^{\pm}, \;\; U_2 P_3^{\pm} U_2^{-1} = P_1^{\pm}, \;\; U_3 P_1^{\pm} U_3^{-1} = P_2^{\pm}. \tag{4.18}$$

We may enhance the similarity transformation to act, formally but trivially, also on the spin operator $\boldsymbol{\sigma}$ as follows

$$\tilde{U}_j = \frac{1}{\sqrt{2}}(1_2 - \mathrm{i}\tau_j) \otimes 1_2, \;\; \tilde{U}_j^{-1} = \frac{1}{\sqrt{2}}(1_2 + \mathrm{i}\tau_j) \otimes 1_2. \tag{4.19}$$

Thus we obtain finally that

$$\tilde{U}_1 \mathbf{J}_{\pm\,2} \tilde{U}_1^{-1} = \mathbf{J}_{\pm\,3}, \;\; \tilde{U}_2 \mathbf{J}_{\pm\,3} \tilde{U}_2^{-1} = \mathbf{J}_{\pm\,1}, \;\; \tilde{U}_3 \mathbf{J}_{\pm\,1} \tilde{U}_3^{-1} = \mathbf{J}_{\pm\,2}. \tag{4.20}$$

As the result of this somewhat tedious procedure we find that the three angular momentum relations (4.17) are equivalent and closely related by similarity transformations. Apparently, the fermion constructed this way is coming in three versions. This is in agreement with the empirical result of elementary particle physics that the quarks come in three different “colours” (and the baryons in the SM come as colourless composites of three quarks). The related symmetry group is $SU(3)$, to which the Yang-Mills gauge theory can be applied. The threefold multiplicity in colour originates from, and is facilitated by, the fact that the key group $SU(2)$ in terms of the tau matrices has exactly three generators.

So, the threefold chiral-spin representation (4.16) of the fermion can describe the quark as a colour triplet. It remains to construct the corresponding Gamma matrices and to define the related Clifford algebra. Before we do this in the subsequent paragraphs we may define the chiral spin matrices for the three quarks by inspection of (4.16) as follows

$$\boldsymbol{\sigma}_j^{\pm} = P_j^{\pm} \otimes \boldsymbol{\sigma}. \tag{4.21}$$

They obey $(\boldsymbol{\sigma}_j^{\pm})^2 = 3P_j^{\pm} \otimes 1_2$, and thus by summing up over the plus and minus sign one obtains three times the unit matrix 1_4. Furthermore, the scalar product of them with opposite sign index vanishes due to the projectors involved, i.e., we have $\boldsymbol{\sigma}_j^{+} \cdot \boldsymbol{\sigma}_j^{-} = 0$. The resulting sigma matrices for the three colour indices $j = 1, 2, 3$ are listed in the appendix.

The aim then is to construct various versions of the Clifford algebra for a fermion based on the chiral spin matrices (4.21). Again, we ought to use both chiral spin matrices and employ them on an equal footing in our calculation. This goal can be achieved by a linear combination of them, while both are acting on a complex Minkowski vector. So, we define the new hermitian 8×8-matrix vector operator

$$\boldsymbol{\Sigma}_{1j} = \tau_2 \otimes \boldsymbol{\sigma}_j^{+} + \tau_3 \otimes \boldsymbol{\sigma}_j^{-} = (\tau_2 \otimes P_j^{+} + \tau_3 \otimes P_j^{-}) \otimes \boldsymbol{\sigma}, \tag{4.22}$$

which can be described as a triple tensor product of the particle-antiparticle, left- and right-chiral, and spin-up and -down doublets. Consequently, $\boldsymbol{\Sigma}_{1j} \cdot \boldsymbol{\Sigma}_{1j} = 3\, 1_8$.

We emphasize again that $\boldsymbol{\Sigma}_{1j}$ has the same metric properties as the Pauli matrices, and thus as the fermion matrices (4.9) based on the four-vector representation of the LG. This metric condition is prerequisite for the validity of the Clifford algebra (3.2). It is worth stressing that the degrees of freedom associated with chirality and particle type become entangled via the definition (4.22).

We omit the colour index j in what follows. We can now define the Gamma matrices for the Dirac equation by tensor multiplication in the following way

$$\Gamma_0 = \tau_1 \otimes 1_4, \quad \boldsymbol{\Gamma} = \mathrm{i}\boldsymbol{\Sigma}_1. \tag{4.23}$$

According to the derivations in the appendix we can after a unitary transformation rewrite this as

$$\Sigma_{01} = \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_1 \end{pmatrix} \otimes 1_2, \;\; \boldsymbol{\Sigma}_1 = \begin{pmatrix} \tau_2 & 0 \\ 0 & \tau_3 \end{pmatrix} \otimes \boldsymbol{\sigma}. \tag{4.24}$$

Close inspection of equation (4.24) reveals the permutation symmetry, namely we can permute the indices of the tau matrices without changing the physical content. As there are only three $SU(2)$ generators, we obtain a threefold multiplicity of the constructed representations, and consequently a triple of Gamma matrices describing the quark. It is apparently coming in six flavours or three generations. This multiplicity originates essentially from the triple chirality-particle-spin as expressed by (4.16) and (4.22), i.e., it is enabled by the notion that the quark (of any colour) exists as a chiral-spin doublet. The Gamma matrices of the Clifford algebra for the flavours are given explicitly in the appendix.

Are the three (or six) flavour states physically equivalent and perhaps connected by a similarity transformation? To answer this question we used again (4.3) and (4.4), but we did not succeed in finding an appropriate similarity transformation. Apparently, flavour is not related to the $SU(3)$ symmetry, and cannot be gauged or described by Yang-Mills theory.

5 DISCUSSION AND CONCLUSION

Our analysis indicates that the fermion may have another dual degree of freedom, which is not apparent in the standard Dirac equation but emerges from the use of manifestly four-dimensional generators of the LG, by which chiral symmetry is entirely ensured in our calculations. The resulting doublet is related to the $SU(2)$ group and thus can be gauged. Technically speaking, this doublet can for a free fermion be decomposed into two Dirac spinors. Yet this important relativistic dual trait is missing in the original Dirac equation, because it consists of two Weyl spinors, and thus employs the chiral $(\frac{1}{2}, 0)$ or the chiral $(0, \frac{1}{2})$ irreducible representations of the LG.

When use is made at the outset of the $(\frac{1}{2}, \frac{1}{2})$ reducible representation of the LG, the Pauli spinors are replaced by what we called complex Minkowski vectors, related to the combined chiral and physical spin, and two such spinors are assembled into the 8-component spinor Φ. It turns out that this "blown-up" description of the fermion reveals two permutation symmetries which are intimately linked to the $SU(2)$ symmetry, namely a threefold (due to three generators of $SU(2)$) permutation symmetry, yielding three generations (flavour) and three colours. This procedure naturally produces colour triplicity, given by the three possible versions of the $\mathbf{J}_{\pm}$ generators of the LG, which can be transformed into each other by a similarity transformation. Thus colour represents a permutation symmetry related to $SU(3)$ that can be gauged.

As emphasized in the paper, the various presented representations of the Clifford algebra or gamma matrices are connected by similarity transformations. Therefore, one can make use of just one of these representations as a standard, of which nowadays the most convenient choices are the chiral or Weyl and the Dirac representations. In the case of the SU(2) and SU(3) symmetry groups, the resulting single Lagrangian consequently is just the usual one of the SM, and one does not have to worry about using different gamma matrix representations in Feynman diagrams.

In conclusion, when use is made of the chirally symmetric $(\frac{1}{2}, \frac{1}{2})$ representation of the LG, the resulting fermion is endowed with another dual degree of freedom, which induces the $SU(2)$ symmetry, and furthermore comes as lepton singlet and as quark triplet with three colours, which induces the $SU(3)$ symmetry. Also, the flavour degrees of freedom naturally emerge from permutation symmetry, but are not connected by a similarity transformation. The theoretical consequences of our study require further investigation.

ACKNOWLEDGEMENT

Eckart Marsch gratefully acknowledges the institutional support by the Extraterrestrial Physics Group headed by Robert Wimmer-Schweingruber at the Institute for Experimental and Applied Physics of Kiel University in Germany.

COMPETING INTERESTS

Author has declared that no competing interests exist.

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APPENDIX: IMPORTANT MATRICES

Matrices of the Rotation and Boost Operators in Minkowski Space

We quote here for completeness the component matrices for the rotation operator $\mathbf{J}$, which read

$$J_\mathrm{x} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathrm{i} \\ 0 & 0 & \mathrm{i} & 0 \end{pmatrix}, J_\mathrm{y} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{i} \\ 0 & 0 & 0 & 0 \\ 0 & -\mathrm{i} & 0 & 0 \end{pmatrix}, J_\mathrm{z} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} & 0 \\ 0 & \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{5.1}$$

The component matrices of the boost vector operator $\mathbf{K}$ are

$$K_\mathrm{x} = \begin{pmatrix} 0 & -\mathrm{i} & 0 & 0 \\ -\mathrm{i} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, K_\mathrm{y} = \begin{pmatrix} 0 & 0 & -\mathrm{i} & 0 \\ 0 & 0 & 0 & 0 \\ -\mathrm{i} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, K_\mathrm{z} = \begin{pmatrix} 0 & 0 & 0 & -\mathrm{i} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\mathrm{i} & 0 & 0 & 0 \end{pmatrix}. \tag{5.2}$$

These formulas can be found in the textbook of [4], for example.

Chiral Spin Matrices for the Quarks

After equations (4.15) and (4.21) the chiral spin matrices were defined as

$$\sigma_j^{\pm} = \frac{1}{2}(1_2 \pm \tau_j) \otimes \sigma. \tag{5.3}$$

By insertion of the tau matrices after (4.1) we thus obtain for the three colours the results

$$\boldsymbol{\sigma}_1^{\pm} = \frac{1}{2}\begin{pmatrix} \boldsymbol{\sigma} & \pm\boldsymbol{\sigma} \\ \pm\boldsymbol{\sigma} & \boldsymbol{\sigma} \end{pmatrix}, \tag{5.4}$$

$$\boldsymbol{\sigma}_2^{\pm} = \frac{1}{2}\begin{pmatrix} \boldsymbol{\sigma} & \mp\mathrm{i}\boldsymbol{\sigma} \\ \pm\mathrm{i}\boldsymbol{\sigma} & \boldsymbol{\sigma} \end{pmatrix}, \tag{5.5}$$

and finally

$$\boldsymbol{\sigma}_3^{+} = \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & 0 \end{pmatrix}, \quad \boldsymbol{\sigma}_3^{-} = \begin{pmatrix} 0 & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}. \tag{5.6}$$

Therefore we obtain, $(\boldsymbol{\sigma}_j^{+})^2 + (\boldsymbol{\sigma}_j^{-})^2 = 3\,1_4$. Let us just consider the last colour index $j = 3$. The associated flavour matrices defining the gamma matrix vector after (4.24) read

$$\boldsymbol{\Sigma}_1 = \tau_2 \otimes \boldsymbol{\sigma}_3^{+} + \tau_3 \otimes \boldsymbol{\sigma}_3^{-} = \tilde{\Sigma}_1 \otimes \boldsymbol{\sigma}, \tag{5.7}$$

$$\boldsymbol{\Sigma}_2 = \tau_3 \otimes \boldsymbol{\sigma}_3^{+} + \tau_1 \otimes \boldsymbol{\sigma}_3^{-} = \tilde{\Sigma}_2 \otimes \boldsymbol{\sigma}, \tag{5.8}$$

$$\boldsymbol{\Sigma}_3 = \tau_1 \otimes \boldsymbol{\sigma}_3^{+} + \tau_2 \otimes \boldsymbol{\sigma}_3^{-} = \tilde{\Sigma}_3 \otimes \boldsymbol{\sigma}. \tag{5.9}$$

The three resulting hermitian 4×4 matrices, which combine the chiral spin and the particle type, then have the following simple form

$$\tilde{\Sigma}_1 = \begin{pmatrix} 0 & 0 & -\mathrm{i} & 0 \\ 0 & 1 & 0 & 0 \\ \mathrm{i} & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tilde{\Sigma}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \tilde{\Sigma}_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\mathrm{i} \\ 1 & 0 & 0 & 0 \\ 0 & \mathrm{i} & 0 & 0 \end{pmatrix} \tag{5.10}$$

Their squares all give the unit matrix 1_4, and they obey cyclically the relation $\tilde{\Sigma_1}\tilde{\Sigma}_2 = \mathrm{i}\tilde{\Sigma}_3$, and thus have all the properties of the Pauli matrices. By help of the unitary matrix

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{5.11}$$

they can therefore be brought into block-diagonal form reading finally

$$\tilde{\Sigma}_1 = \begin{pmatrix} \tau_2 & 0 \\ 0 & \tau_3 \end{pmatrix}, \ \tilde{\Sigma}_2 = \begin{pmatrix} \tau_3 & 0 \\ 0 & \tau_1 \end{pmatrix}, \tilde{\Sigma}_3 = \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix}. \tag{5.12}$$

Similarly we obtain for the Σ_{0j} matrices

$$\tilde{\Sigma}_{01} = \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_1 \end{pmatrix}, \ \tilde{\Sigma}_{02} = \begin{pmatrix} \tau_2 & 0 \\ 0 & \tau_2 \end{pmatrix}, \tilde{\Sigma}_{03} = \begin{pmatrix} \tau_3 & 0 \\ 0 & \tau_3 \end{pmatrix}. \tag{5.13}$$

These equations exactly correspond to the six ones of (4.5) for the standard Dirac equation, whereby the previously only assumed doublets now appear naturally as chiral-spin related doublets. The corresponding Gamma matrices for the three generations (with index $f = 1, 2, 3$) read

$$\Gamma_{0f} = \tilde{\Sigma}_{0f} \otimes 1_2; \ \ \mathbf{\Gamma}_f = \mathrm{i}\tilde{\Sigma}_f \otimes \boldsymbol{\sigma}. \tag{5.14}$$

These Gammas obey the Clifford algebra (3.2). As the chiral spins are equivalent for the three colours due to the similarity transformations after (4.17), we can choose (5.14) as the standard representation of the Gammas for all three colours.

APPENDIX: LORENTZ TRANSFORMATION OF THE DIRAC SPINOR AND SOLUTION OF THE DIRAC WAVE EQUATION

In completion of the discussion of the Dirac equation in terms of the abstract Clifford algebra, we introduce here the hermitian spin (rotation) operator and antihermitian rapidity (boost) operator associated with the spinor wave equation (3.4) as follows

$$\mathbf{S} = \frac{\mathrm{i}}{4}(\mathbf{\Gamma} \times \mathbf{\Gamma}), \quad \mathbf{R} = \frac{\mathrm{i}}{2}\Gamma_0\mathbf{\Gamma}. \tag{5.15}$$

According to their definitions, the spin and rapidity operators obey the linked three-vector equations of the Lorentz algebra, i.e., we have $\mathbf{S} \times \mathbf{S} = \mathrm{i}\mathbf{S}$, $\mathbf{R} \times \mathbf{R} = -\mathrm{i}\mathbf{S}$, $\mathbf{S} \times \mathbf{R} = \mathbf{R} \times \mathbf{S} = \mathrm{i}\mathbf{R}$. This can be shown by straightforward application of the rules given by the Clifford algebra (3.2). Moreover, $\mathbf{S}^2 = \frac{3}{4}$ as expected for a fermion. We define like in (2.2) the right- and left-chiral spin and find that

$$\mathbf{S}_\pm = \frac{1}{2}(\mathbf{S} \pm \mathrm{i}\mathbf{R}) = \mathbf{S}P_\mp, \tag{5.16}$$

involving the projection operators. The associated spinorial LT named Λ acts on the spinor Ψ and can after some algebra (exploiting the properties of the projection operator) be written

$$\Lambda = \exp{(\mathrm{i}\boldsymbol{\theta} \cdot \mathbf{S} + \mathrm{i}\boldsymbol{\beta} \cdot \mathbf{R})} = \exp{(\mathrm{i}\boldsymbol{\theta}_+ \cdot \mathbf{S})}P_- + \exp{(\mathrm{i}\boldsymbol{\theta}_- \cdot \mathbf{S})}P_+\,, \tag{5.17}$$

with the complex angle $\boldsymbol{\theta}_\pm = \boldsymbol{\theta} \pm \mathrm{i}\boldsymbol{\beta}$. Because of this complex angle, the LT is not a hermitian operator.

The linear wave equation (3.4) has two fundamental plane wave solution with positive (antiparticle) and negative (particle) frequency. We make for the particle and its antiparticle the standard plane wave ansatz

$$\Psi(t, \mathbf{x}) = U_{\pm}(\mathbf{p}) \exp\left(\mp \mathrm{i}(E(p)t - \mathbf{p} \cdot \mathbf{x})\right). \tag{5.18}$$

Here the positive relativistic energy reads $E(p) = \sqrt{m^2 + p^2}$. Thus we obtain the algebraic equation for the two related polarization spinors, reading

$$(\Gamma^{\mu} p_{\mu} \mp m) U_{\pm}(\mathbf{p}) = 0, \tag{5.19}$$

which just differ by the sign in front of the mass. Here $p_{\mu} = (E(p), -\mathbf{p})$ is the covariant four momentum. To calculate explicitly the polarization spinors $U_{\pm}(\mathbf{p})$ we need to know the detailed form of the gamma matrices. When operating with Γ_5 on the wave equation (3.4) or its Fourier version (5.19), we find that $\Gamma\Psi$ solves it for a negative sign at the mass. So, we may also call the operator Γ_5 the mass conjugation operator.

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