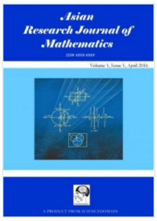

Asian Research Journal of Mathematics

12(2): 1-12, 2019; Article no.ARJOM.47220

ISSN: 2456-477X

Analytical Study of the Difference Equation

$$x_{n+1} = \frac{x_n x_{n-6}}{x_{n-5}\left(\pm 1 - x_n x_{n-6}\right)}$$

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Authors' contributions

This work was carried out in collaboration between all authors. All authors read and approved the final manuscript.

Article Information

DOI: 10.9734/ARJOM/2019/v12i230081

Editor(s):

(1) Dr. Krasimir Yankov Yordzhev, Associate Professor, Faculty of Mathematics and Natural Sciences, South-West University, Blagoevgrad, Bulgaria.

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(3) Abdullah Sonmezoglu, Bozok University, Turkey.

Complete Peer review History: http://www.sdiarticle3.com/review-history/47220

Received: 20 November 2018

Accepted: 02 February 2019

Published: 16 February 2019

Original Research Article

Abstract

This paper is devoted to find the form of the solutions of a rational difference equations with arbitrary positive real initial conditions. Specific form of the solutions of two special cases of this equation are given.

Keywords: Difference equations; Recursive sequences; Stability; Boundedness.

2010 Mathematics Subject Classification: 39A10.

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1 Introduction

Our aim in this paper is to investigate the behavior of the solution of the following nonlinear difference equation

$$x_{n+1} = \frac{x_n x_{n-6}}{x_{n-5}\left(\pm 1 - x_n x_{n-6}\right)}, \quad n = 0, 1, ..., \tag{1.1}$$

where the initial conditions $x_{-6}, x_{-5}, x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_0$ are arbitrary positive real numbers.

The study and solution of nonlinear rational recursive sequence of high order is quite challenging and rewarding [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13]. Recently, there has been a lot of interest in studying the qualitative properties of rational recursive sequences, Furthermore diverse nonlinear trend occurring in science and engineering can be modeled by such equations and the solution about such equations offer prototypes towards the development of the theory [14, 15, 16, 17, 18, 19, 20, 21, 22]. However, there have not been any suitable general method to deal with the global behavior of rational difference equations of high order so far [23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56]. Therefore, the study of rational difference equations of order greater than one is worth further consideration.

Many researchers have investigated the behavior of the solution of difference equations, for example,

Elsayed et al. [33] has obtained results concerning the dynamics and solution of the rational difference equation

$$x_{n+1} = \frac{x_{n-1} x_{n-5}}{x_{n-3}\left(\pm 1 \pm x_{n-1} x_{n-5}\right)}.$$

Elsayed et al. [38] has obtained results concerning The dynamics and the solutions of the rational difference equation

$$x_{n+1} = \frac{x_n x_{n-4}}{x_{n-3}\left(\pm 1 \pm x_n x_{n-4}\right)}.$$

Aloqeili [3] has obtained the solutions of the difference equation

$$x_{n+1} = \frac{x_{n-1}}{a - x_n x_{n-1}}.$$

Simsek et al. [48] obtained the solution of the difference equation

$$x_{n+1} = \frac{x_{n-3}}{1 + x_{n-1}}$$

Çinar [7, 8, 9] got the solutions of the following difference equation

$$x_{n+1} = \frac{x_{n-1}}{1 + a x_n x_{n-1}}, x_{n+1} = \frac{x_{n-1}}{-1 + a x_n x_{n-1}}, x_{n+1} = \frac{a x_{n-1}}{1 + b x_n x_{n-1}}$$

In [39], Ibrahim got the form of the solution of the rational difference equation

$$x_{n+1} = \frac{x_n x_{n-2}}{x_{n-1}(a + b x_n x_{n-2})}$$

Karatas et al. [41] got the solution of the difference equation

$$x_{n+1} = \frac{x_{n-5}}{1 + x_{n-2} x_{n-5}}$$

Here, we recall some notations and results which will be useful in our investigation. Let I be some interval of real numbers and let

$$f : I^{k+1} \to I,$$

be a continuously differentiable function. Then for every set of initial conditions $x_{-k}, x_{-k+1}, x_{-k+2}, ..., x_0 \in I$, the difference equation

$$x_{n+1} = f(x_n, x_{n-1}, ..., x_{n-k}), \quad n = 0, 1, ..., \tag{1.2}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$.

Definition 1. (Equilibrium Point) A point $\bar{x} \in I$ is called an equilibrium point of Eq. (1.2) if $\bar{x} = f(\bar{x}, \bar{x}, ..., \bar{x})$. That is, $x_n = \bar{x}$ for $n \geq 0$, is a solution of Eq. (1.2), or equivalently, $\bar{x}$ is a fixed point of f.

Definition 2. (Stability)

- The equilibrium point $\bar{x}$ of Eq. (1.2) is locally stable if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x_{-k}, x_{-k+1}, x_{-k+2}, ..., x_0 \in I$, with

  $$|x_{-k} - \bar{x}| + |x_{-k+1} - \bar{x}| + |x_{-k+2} - \bar{x}| + ... + |x_0 - \bar{x}| < \delta,$$

  we have $|x_n - \bar{x}| < \varepsilon$, for all $n \geq -k$.
- The equilibrium point $\bar{x}$ of Eq. (1.2) is locally asymptotically stable if $\bar{x}$ is locally stable solution of Eq. (1.2) and there exists $\gamma > 0$, such that for all $x_{-k}, x_{-k+1}, x_{-k+2}, ..., x_0 \in I$, with

  $$|x_{-k} - \bar{x}| + |x_{-k+1} - \bar{x}| + |x_{-k+2} - \bar{x}| + ... + |x_0 - \bar{x}| < \delta,$$

  we have $\lim_{n\to\infty} x_n = \bar{x}$.
- The equilibrium point $\bar{x}$ of Eq. (1.2) is global attractor if for all $x_{-k}, x_{-k+1}, ..., x_0 \in I$ we have

  $$\lim_{n\to\infty} x_n = \bar{x}.$$
- The equilibrium point $\bar{x}$ of Eq. (1.2) is globally asymptotically stable if $\bar{x}$ is locally stable, and $\bar{x}$ is also a global attractor of Eq. (1.2).
- The equilibrium point $\bar{x}$ of Eq. (1.2) is unstable if $\bar{x}$ is not locally stable.

The linearized equation of Eq. (1.2) about the equilibrium $\bar{x}$ is the linear difference equation

$$y_{n+1} = \sum_{i=0}^{k} \frac{\partial f(\bar{x}, \bar{x}, ..., \bar{x})}{\partial x_{n-i}} y_{n-i}$$

Theorem 1. Assume that $p, q \in \mathbb{R}$ and $k \in \{0, 1, 2, ...\}$. Then $|p| + |q| < 1$ is a sufficient condition for the asymptotic stability of the difference equation

$$x_{n+1} + px_n + qx_{n-k} = 0, n = 0, 1, ...$$

Remark 1. The theorem can be easily extended to a general linear equations of the form

$$x_{n+k} + p_1 x_{n+k-1} + ... + p_k x_n = 0, \quad n = 0, 1, ..., \tag{1.3}$$

where $p_1, p_2, ..., p_k \in \mathbb{R}$ and $k \in \{0, 1, 2, ...\}$. Then Eq. (1.3) is asymptotically stable provided that $\sum_{i=0}^{k} |p_i| < 1$.

Our goal in this section is to find a specific form of the solutions of some special cases of Eq. (1.1) and give numerical examples of each case.

2 On the Difference Equation $x_{n+1} = \dfrac{x_n x_{n-6}}{x_{n-5}\left(1 - x_n x_{n-6}\right)}$

In this subsection we study the following special case of Eq. (1.1):

$$x_{n+1} = \frac{x_n x_{n-6}}{x_{n-5}\left(1 - x_n x_{n-6}\right)}, \quad n = 0, 1, ..., \tag{2.1}$$

where the initial conditions $x_{-6}, x_{-5}, x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_0$ are arbitrary nonzeros real numbers.

Theorem 2. Let $\{x_n\}_{n=-4}^{\infty}$ be a solution of Eq. (2.1). Then for $n = 0, 1, ...,$

$$x_{12n-6} = x_{-6}\prod_{i=0}^{n-1}\left(\frac{1-(12i)x_{-6}x_0}{1-(12i+6)x_{-6}x_0}\right), \qquad x_{12n-5} = x_{-5}\prod_{i=0}^{n-1}\left(\frac{1-(12i+1)x_{-6}x_0}{1-(12i+7)x_{-6}x_0}\right),$$
$$x_{12n-4} = x_{-4}\prod_{i=0}^{n-1}\left(\frac{1-(12i+2)x_{-6}x_0}{1-(12i+8)x_{-6}x_0}\right), \qquad x_{12n-3} = x_{-3}\prod_{i=0}^{n-1}\left(\frac{1-(12i+3)x_{-6}x_0}{1-(12i+9)x_{-6}x_0}\right),$$
$$x_{12n-2} = x_{-2}\prod_{i=0}^{n-1}\left(\frac{1-(12i+4)x_{-6}x_0}{1-(12i+10)x_{-6}x_0}\right), \qquad x_{12n-1} = x_{-1}\prod_{i=0}^{n-1}\left(\frac{1-(12i+5)x_{-6}x_0}{1-(12i+11)x_{-6}x_0}\right),$$
$$x_{12n} = x_0\prod_{i=0}^{n-1}\left(\frac{1-(12i+6)x_{-6}x_0}{1-(12i+12)x_{-6}x_0}\right), \qquad x_{12n+1} = \frac{x_{-6}x_0}{x_{-5}(1-x_{-6}x_0)}\prod_{i=0}^{n-1}\left(\frac{1-(12i+7)x_{-6}x_0}{1-(12i+13)x_{-6}x_0}\right),$$
$$x_{12n+2} = \frac{x_{-6}x_0}{x_{-4}(1-2x_{-6}x_0)}\prod_{i=0}^{n-1}\left(\frac{1-(12i+8)x_{-6}x_0}{1-(12i+14)x_{-6}x_0}\right), \qquad x_{12n+3} = \frac{x_{-6}x_0}{x_{-3}(1-3x_{-6}x_0)}\prod_{i=0}^{n-1}\left(\frac{1-(12i+9)x_{-6}x_0}{1-(12i+15)x_{-6}x_0}\right),$$
$$x_{12n+4} = \frac{x_{-6}x_0}{x_{-2}(1-4x_{-6}x_0)}\prod_{i=0}^{n-1}\left(\frac{1-(12i+10)x_{-6}x_0}{1-(12i+16)x_{-6}x_0}\right), \qquad x_{12n+5} = \frac{x_{-6}x_0}{x_{-1}(1-5x_{-6}x_0)}\prod_{i=0}^{n-1}\left(\frac{1-(12i+11)x_{-6}x_0}{1-(12i+17)x_{-6}x_0}\right).$$

Proof: We use an inductive proof for this rational recursive sequences. It is easy to see that for $n = 0$, the result holds. Suppose that $n > 0$ and that the assumption is satisfied for $n - 1$. That is;

$$x_{12n-18} = x_{-6}\prod_{i=0}^{n-2}\left(\frac{1-(12i)x_{-6}x_0}{1-(12i+6)x_{-6}x_0}\right), \qquad x_{12n-17} = x_{-5}\prod_{i=0}^{n-2}\left(\frac{1-(12i+1)x_{-6}x_0}{1-(12i+7)x_{-6}x_0}\right),$$
$$x_{12n-16} = x_{-4}\prod_{i=0}^{n-2}\left(\frac{1-(12i+2)x_{-6}x_0}{1-(12i+8)x_{-6}x_0}\right), \qquad x_{12n-15} = x_{-3}\prod_{i=0}^{n-2}\left(\frac{1-(12i+3)x_{-6}x_0}{1-(12i+9)x_{-6}x_0}\right),$$
$$x_{12n-14} = x_{-2}\prod_{i=0}^{n-2}\left(\frac{1-(12i+4)x_{-6}x_0}{1-(12i+10)x_{-6}x_0}\right), \qquad x_{12n-13} = x_{-1}\prod_{i=0}^{n-2}\left(\frac{1-(12i+5)x_{-6}x_0}{1-(12i+11)x_{-6}x_0}\right),$$
$$x_{12n-12} = x_0\prod_{i=0}^{n-2}\left(\frac{1-(12i+6)x_{-6}x_0}{1-(12i+12)x_{-6}x_0}\right), \qquad x_{12n-11} = \frac{x_{-6}x_0}{x_{-5}(1-x_{-6}x_0)}\prod_{i=0}^{n-2}\left(\frac{1-(12i+7)x_{-6}x_0}{1-(12i+13)x_{-6}x_0}\right),$$
$$x_{12-10} = \frac{x_{-6}x_0}{x_{-4}(1-2x_{-6}x_0)}\prod_{i=0}^{n-2}\left(\frac{1-(12i+8)x_{-6}x_0}{1-(12i+14)x_{-6}x_0}\right), \qquad x_{12n-9} = \frac{x_{-6}x_0}{x_{-3}(1-3x_{-6}x_0)}\prod_{i=0}^{n-2}\left(\frac{1-(12i+9)x_{-6}x_0}{1-(12i+15)x_{-6}x_0}\right),$$
$$x_{12n-8} = \frac{x_{-6}x_0}{x_{-2}(1-4x_{-6}x_0)}\prod_{i=0}^{n-2}\left(\frac{1-(12i+10)x_{-6}x_0}{1-(12i+16)x_{-6}x_0}\right), \qquad x_{12n-7} = \frac{x_{-6}x_0}{x_{-1}(1-5x_{-6}x_0)}\prod_{i=0}^{n-2}\left(\frac{1-(12i+11)x_{-6}x_0}{1-(12i+17)x_{-6}x_0}\right).$$

Now, using the main Eq. (2.1), one has

$$x_{12n-6} = \frac{x_{12n-7}x_{12n-13}}{x_{12n-12}\left(1 - x_{12n-7}x_{12n-13}\right)}$$

$$= \frac{\frac{x_{-6}x_0}{x_{-1}(1-5x_{-6}x_0)}\prod_{i=0}^{n-2}(\frac{1-(12i+11)x_{-6}x_0}{1-(12i+17)x_{-6}x_0})x_{-1}\prod_{i=0}^{n-2}(\frac{1-(12i+5)x_{-6}x_0}{1-(12i+11)x_{-6}x_0})}{x_0\prod_{i=0}^{n-2}(\frac{1-(12i+6)x_{-6}x_0}{1-(12i+12)x_{-6}x_0})\left(1-\frac{x_{-6}x_0}{x_{-1}(1-5x_{-6}x_0)}\prod_{i=0}^{n-2}(\frac{1-(12i+11)x_{-6}x_0}{1-(12i+17)x_{-6}x_0})x_{-1}\prod_{i=0}^{n-2}(\frac{1-(12i+5)x_{-6}x_0}{1-(12i+11)x_{-6}x_0})\right)}$$

$$= \frac{\frac{x_{-6}}{(1-5x_{-6}x_0)}\prod_{i=0}^{n-2}(\frac{1-(12i+5)x_{-6}x_0}{1-(12i+17)x_{-6}x_0})}{\prod_{i=0}^{n-2}(\frac{1-(12i+6)x_{-6}x_0}{1-(12i+12)x_{-6}x_0})\left(1-\frac{x_{-6}x_0}{(1-5x_{-6}x_0)}\prod_{i=0}^{n-2}(\frac{1-(12i+5)x_{-6}x_0}{1-(12i+17)x_{-6}x_0})\right)}$$

$$= \prod_{i=0}^{n-2}(\frac{1-(12i+12)x_{-6}x_0}{1-(12i+6)x_{-6}x_0})\frac{\frac{x_{-6}}{(1-5x_{-6}x_0)}\prod_{i=0}^{n-2}(\frac{1-(12i+5)x_{-6}x_0}{1-(12i+17)x_{-6}x_0})}{\left(1-\frac{x_{-6}x_0}{(1-5x_{-6}x_0)}\prod_{i=0}^{n-2}(\frac{1-(12i+5)x_{-6}x_0}{1-(12i+17)x_{-6}x_0})\right)}$$

$$= \prod_{i=0}^{n-2}(\frac{1-(12i+12)x_{-6}x_0}{1-(12i+6)x_{-6}x_0})\frac{x_{-6}\frac{1}{1-(12n-7)x_{-6}x_0}}{\left(1-\frac{x_{-6}x_0}{(1-(12n-7)x_{-6}x_0})\right)}$$

$$= \prod_{i=0}^{n-2}(\frac{1-(12i+12)x_{-6}x_0}{1-(12i+6)x_{-6}x_0})\frac{x_{-6}}{1-(12n-6)x_{-6}x_0}.$$

Hence, we have

$$x_{12n-6} = x_{-6}\prod_{i=0}^{n-1}(\frac{1-(12i)x_{-6}x_0}{1-(12i+6)x_{-6}x_0}).$$

Similarly, using the main Eq. (2.1), one has

$$x_{12n-5} = \frac{x_{12n-6}x_{12n-12}}{x_{12n-11}\left(1 - x_{12n-6}x_{12n-12}\right)}$$

$$= \frac{x_{-6}\prod_{i=0}^{n-1}(\frac{1-(12i)x_{-6}x_0}{1-(12i+6)x_{-6}x_0})x_0\prod_{i=0}^{n-2}(\frac{1-(12i+6)x_{-6}x_0}{1-(12i+12)x_{-6}x_0})}{\frac{x_{-6}x_0}{x_{-5}(1-x_{-6}x_0)}\prod_{i=0}^{n-2}(\frac{1-(12i+7)x_{-6}x_0}{1-(12i+13)x_{-6}x_0})\left(1-x_{-6}\prod_{i=0}^{n-1}(\frac{1-(12i)x_{-6}x_0}{1-(12i+6)x_{-6}x_0})x_0\prod_{i=0}^{n-2}(\frac{1-(12i+6)x_{-6}x_0}{1-(12i+12)x_{-6}x_0})\right)}$$

$$= x_{-5}(1+x_{-6}x_0)\prod_{i=0}^{n-2}(\frac{1-(12i+13)x_{-6}x_0}{1-(12i+7)x_{-6}x_0})\frac{(\frac{1-(12n-12)x_{-6}x_0}{1-(12n-6)x_{-6}x_0})\prod_{i=0}^{n-2}(\frac{1-(12i)x_{-6}x_0}{1-(12i+12)x_{-6}x_0})}{\left(1-x_{-6}x_0(\frac{1-(12n-12)x_{-6}x_0}{1-(12n-6)x_{-6}x_0})\prod_{i=0}^{n-2}(\frac{1-(12i)x_{-6}x_0}{1-(12i+12)x_{-6}x_0})\right)}$$

$$= x_{-5}\prod_{i=0}^{n-2}(\frac{1-(12i+1)x_{-6}x_0}{1-(12i+7)x_{-6}x_0})\frac{1-x_{-6}x_0}{1-(12n-5)x_{-6}x_0}.$$

Hence, we have

$$x_{12n-5} = x_{-5} \prod_{i=0}^{n-1} \left(\frac{1-(12i+1)x_{-6}x_0}{1-(12i+7)x_{-6}x_0} \right).$$

Similarly, one can easily obtain the other relations. Thus, the proof is completed.

Theorem 3. Eq. (2.1) has one equilibrium point $\bar{x} = 0$ and this equilibrium point is not locally asymptotically stable.

Proof. In this section we investigate the local stability character of the solutions of Eq. (2.1). Equation (2.1) has a unique positive equilibrium point and is given by

$$\bar{x} = \frac{\bar{x}^2}{\bar{x}\left(1-\bar{x}^2\right)} = \frac{\bar{x}}{1-\bar{x}^2}, \quad \text{or also} \quad 1 = 1-\bar{x}^2,$$

then the unique equilibrium point is given by $\bar{x} = 0$.

Define the following function

$$f : (0,\infty)^3 \to (0,\infty)$$
$$f(u,v,w) = \frac{uw}{v\left(1-uw\right)}.$$

Therefore it follows that

$$f_u(u,v,w) = \frac{w}{v\left(1-uw\right)^2}, \quad f_v(u,v,w) = -\frac{uw}{v^2\left(1-uw\right)}, \quad f_w(u,v,w) = \frac{u}{v\left(1-uw\right)^2}.$$

Then

$$f_u(\bar{x},\bar{x},\bar{x}) = \frac{1}{\left(1-\bar{x}^2\right)^2} = 1, \quad f_v(\bar{x},\bar{x},\bar{x}) = -\frac{1}{\left(1-\bar{x}^2\right)} = -1, \quad f_w(\bar{x},\bar{x},\bar{x}) = \frac{1}{\left(1-\bar{x}^2\right)^2} = 1.$$

The linearized equation of Eq. (2.1) about $\bar{x}$ is

$$y_{n+1} - y_{n-6} + y_{n-5} - y_n = 0. \tag{2.2}$$

It follows from Theorem 1 that Eq. (2.2) is not asymptotically stable. The proof is complete.

For confirming the results of this section, we consider numerical example for ,(See Fig. 1).

3 On the Difference Equation $x_{n+1} = \dfrac{x_n x_{n-6}}{x_{n-5}\left(-1 - x_n x_{n-6}\right)}$

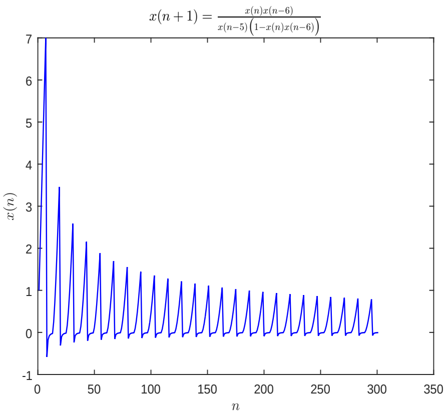
In this subsection we study the following special case of Eq. (1.1):

$$x_{n+1} = \frac{x_n x_{n-6}}{x_{n-5}\left(-1 - x_n x_{n-6}\right)}, \quad n = 0, 1, ..., \tag{3.1}$$

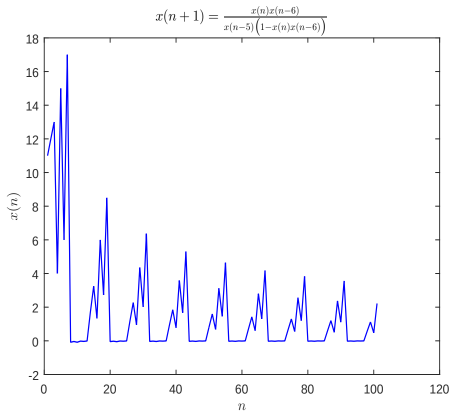
where the initial conditions $x_{-6}, x_{-5}, x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_0$ are arbitrary nonzeros real numbers.

Theorem 4. Let $\{x_n\}_{n=-4}^{\infty}$ be a solution of Eq. (3.1). Then the solution of Eq. (3.1) is bounded and periodic of period 12 given by:

$$x_{12n-6} = x_{-6}, \quad x_{12n-5} = x_{-5}, \quad x_{12n-4} = x_{-4}, \quad x_{12n-3} = x_{-3},$$
$$x_{12n-2} = x_{-2}, \quad x_{12n-1} = x_{-1}, \quad x_{12n} = x_0, \quad x_{12n+1} = \frac{x_0 x_{-6}}{x_{-5}(-1 - x_0 x_{-6})},$$
$$x_{12n+2} = \frac{x_0 x_{-6}}{x_{-4}}, \quad x_{12n+3} = \frac{x_0 x_{-6}}{x_{-3}(-1 - x_0 x_{-6})}, \quad x_{12n+4} = \frac{x_0 x_{-6}}{x_{-2}}, \quad x_{12n+5} = \frac{x_0 x_{-6}}{x_{-1}(-1 - x_0 x_{-6})}.$$



(a) $x_{-6} = 1, x_{-5} = 2, x_{-4} = 3, x_{-3} = 4, x_{-2} = 5, x_{-1} = 6, x_0 = 7$.



(b) $x_{-6} = 11, x_{-5} = 12, x_{-4} = 13, x_{-3} = 4, x_{-2} = 15, x_{-1} = 6, x_0 = 17$.

Fig. 1: Behavior of the solution of system (2.1). It can be seen that the solution does't converge to zero wich confirm the fact that the equilibrium point 0 is not locally asymptotically stable.

Proof. For $n = 0$, the result holds. Now suppose that our assumption holds for $n - 1$. That is;

$$x_{12n-18} = x_{-6}, \quad x_{12n-17} = x_{-5}, \quad x_{12n-16} = x_{-4}, \quad x_{12n-15} = x_{-3},$$
$$x_{12n-14} = x_{-2}, \quad x_{12n-13} = x_{-1}, \quad x_{12n-12} = x_0, \quad x_{12n-11} = \frac{x_0 x_{-6}}{x_{-5}(-1 - x_0 x_{-6})},$$
$$x_{12n-10} = \frac{x_0 x_{-6}}{x_{-4}}, \quad x_{12n-9} = \frac{x_0 x_{-6}}{x_{-3}(-1 - x_0 x_{-6})}, \quad x_{12n-8} = \frac{x_0 x_{-6}}{x_{-2}}, \quad x_{12n-7} = \frac{x_0 x_{-6}}{x_{-1}(-1 - x_0 x_{-6})}.$$

Now it follows from Eq. (3.1) that

$$x_{12n-6} = \frac{x_{12n-7}x_{12n-13}}{x_{12n-12}\left(-1 - x_{12n-7}x_{12n-13}\right)} = \frac{\frac{x_0 x_{-6}}{x_{-1}(-1 - x_0 x_{-6})}x_{-1}}{x_0\left(-1 - \frac{x_0 x_{-6}}{x_{-1}(-1 - x_0 x_{-6})}x_{-1}\right)}$$
$$= \frac{\frac{x_{-6}}{(-1 - x_0 x_{-6})}}{\left(-1 - \frac{x_0 x_{-6}}{(-1 - x_0 x_{-6})}\right)} = \frac{\frac{x_{-6}}{(-1 - x_0 x_{-6})}}{\left(\frac{1}{(-1 - x_0 x_{-6})}\right)} = x_{-6}$$

Similarly

$$x_{12n-5} = \frac{x_{12n-6}x_{12n-12}}{x_{12n-11}\left(-1 - x_{12n-6}x_{12n-12}\right)} = \frac{x_{-6}x_0}{\frac{x_0 x_{-6}}{x_{-5}(-1 - x_0 x_{-6})}\left(-1 - x_{-6}x_0\right)} = x_{-5}$$

$$x_{12n-4} = \frac{x_{12n-5}x_{12n-11}}{x_{12n-10}\left(-1 - x_{12n-5}x_{12n-11}\right)} = \frac{x_{-5}\dfrac{x_0x_{-6}}{x_{-5}(-1-x_0x_{-6})}}{\dfrac{x_0x_{-6}}{x_{-4}}\left(-1 - x_{-5}\dfrac{x_0x_{-6}}{x_{-5}(-1-x_0x_{-6})}\right)}$$

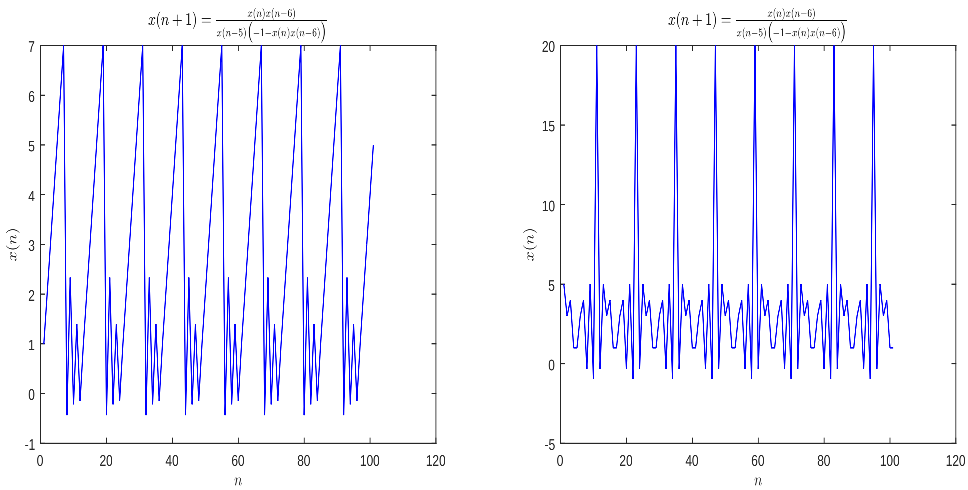
$$= \frac{\dfrac{1}{(-1-x_0x_{-6})}}{\dfrac{1}{x_{-4}}\left(-1 - \dfrac{x_0x_{-6}}{(-1-x_0x_{-6})}\right)} = \frac{\dfrac{x_{-4}}{(-1-x_0x_{-6})}}{\left(\dfrac{1}{(-1-x_0x_{-6})}\right)} = x_{-4}$$

Similarly, one can easily obtain the other relations. Thus, the proof is completed.

Theorem 5. Eq. (3.1) has a unique equilibrium point which is 0 and this equilibrium point is not locally asymptotically stable.

Proof. As the proof of Theorem 3 and will be omitted.

For confirming the results of this section, we consider the following numerical examples,(See Figure 2).



(a) $x_{-6} = 1, x_{-5} = 2, x_{-4} = 3,$
$x_{-3} = 4, x_{-2} = 5, x_{-1} = 6, x_0 = 7.$

(b) $x_{-6} = 5, x_{-5} = 3, x_{-4} = 4,$
$x_{-3} = 1, x_{-2} = 1, x_{-1} = 3, x_0 = 4.$

Fig. 2: Behavior of the solution of system (3.1). It can be seen that the solution does't converge to zero wich confirm the fact that the equilibrium point 0 is not locally asymptotically stable.

4 Conclusion

We investigated, in this paper, the behavior of the solution of the following nonlinear difference equation

$$x_{n+1} = \frac{x_n x_{n-6}}{x_{n-5}\left(\pm 1 - x_n x_{n-6}\right)}, \quad n = 0, 1, ...,$$

with arbitrary positive real initial conditions $x_{-6}, x_{-5}, x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_0$. Local stability is discussed and the expressions of the solution of some special cases are given and validated by some numerical examples.

Acknowledgements

This article was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah. The authors, therefore, acknowledge with thanks DSR technical and financial support.

Competing Interests

Authors have declared that no competing interests exist.

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