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On Riesz Sections in Sequence Spaces

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Authors' contributions

This work was carried out in collaboration between all authors. All authors equally contributed into the study. All authors read and approved the final manuscript.

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Abstract

The theory of FK spaces was introduced by Zeller in [1] and some properties of sectional subspaces in FK spaces were investigated by Zeller in [2]. The notion of Cesàro sections in FK spaces was studied in [3]. In [4], Buntinas examined Toeplitz sections in sequence spaces and characterized some properties. In this paper, we introduce Riesz sections in sequence spaces and examine some properties of them.

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1 Introduction

In recent years, constructing dual pairs of sequence spaces and investigating the properties AK, AB, SAK etc. in FK spaces was used by Boos and Leiger [5–8], Garling [9]. Also, the new technique for deducing certain topological properties, for example, AB, KB, AD properties, solidity and monotonicity etc., and determining the β and γ duals of the domain of a triangle matrix in a sequence space is presented by Altay and Başar [10].

Let ω denote the space of all real or complex–valued sequences. It can be topologized with the seminorms $p_i(x) = |x_i|, \quad (i = 1, 2, ...)$, and any vector subspace of ω is called a sequence space. A sequence space X, with a vector space topology τ is a K space provided that the inclusion mapping $I : (X, \tau) \to \omega, \quad I(x) = x$ is continuous. If, in addition, τ is complete, metrizable and locally convex then (X, τ) is called an FK space. So an FK space is a complete, metrizable local convex topological vector space of sequences for which the coordinate functionals are continuous. An FK space whose topology is normable is called a BK space. The basic properties of such spaces can be found in [11], [12] and [13]. By m, c_0 we denote the space of all bounded sequences, null sequences, respectively. These are FK space under $\|x\| = \sup_n |x_n|$. By ℓ we shall denote the space of all absolutely summable sequences. The sequences space

$$\begin{aligned} cs &= \left\{ x \in \omega : \sum_{j=1}^{\infty} x_j \quad \text{convergent} \right\}, \\ bs &= \left\{ x \in \omega : \sup_n \left| \sum_{j=1}^{n} x_j \right| < \infty \right\}, \\ \rho &= \left\{ \alpha \in \omega : \sum_n a_{nk} \quad \text{convergent and} \quad \sup_m \sum_k \left| \sum_{n=0}^{m} (a_{nk} - a_{nk-1}) \right| < \infty \right\}, \end{aligned}$$

in which $A = (a_{nk}) = R.diag(\alpha_1, \alpha_2, \dots).R^{-1} = R.diag(\alpha).R^{-1}$ and R is Riesz matrix.

$$\begin{aligned} (cs)_R = rs &= \left\{ x \in \omega : \lim_n \left| \frac{1}{Q_n} \sum_{k=1}^{n} \sum_{j=1}^{k} q_j x_j \right| \quad \text{exists} \right\}, \\ (bs)_R = rb &= \left\{ x \in \omega : \sup_n \left| \frac{1}{Q_n} \sum_{k=1}^{n} \sum_{j=1}^{k} q_j x_j \right| < \infty \right\}, \\ (c_0)_R = r_0 &= \left\{ x \in \omega : \lim_n \left| \frac{1}{Q_n} \sum_{j=1}^{n} q_j x_j \right| = 0 \right\} \end{aligned}$$

are FK spaces with the norms

$$\|x\|_{rb} = \sup_n \left| \frac{1}{Q_n} \sum_{k=1}^{n} \sum_{j=1}^{k} q_j x_j \right|,$$

$$\|x\|_{r_o} = \sup_n \frac{1}{Q_n} \left| \sum_{k=1}^{n} q_j x_j \right|$$

respectively. Throughout the paper e denote the sequence of ones, $(1, 1, 1, \dots, 1, \dots); e^j, (j = 1, 2, \dots)$, the sequence $(0, 0, \dots, 1, 0, \dots)$ with the one in the j-th position. Let $\phi = span\{e^k : k \in N\}$ and $\phi_1 = \phi \cup \{e\}$. The topological dual of X is denoted by X'. Let (X, τ) be a K space with $\phi \subset X$ and dual space X', and let $x = (x_k) \in X$ be arbitrarily given. Then

$$x^{[n]} = \sum_{k=1}^{n} x_k e^k = (x_1, x_2, \dots, x_n, 0, \dots)$$

is called the n^{th} section of x. We define the following properties:
x has $AK(sectional\ \ convergence)$ if $x^{[n]} \to x$ in (X, τ).
x has $SAK(weak\ \ sectional\ \ convergence)$ if $x^{[n]} \to x$ in $(X, \sigma(X, X^{'}))$.
x has $FAK(functional\ \ sectional\ \ convergence)$ if $\sum_k x_k f(e^k)$ converges for all $f \in X^{'}$.
x has $AB(sectional\ \ boundedness)$ if $\{x^{[n]} : n \in \mathbb{N}\}$ is bounded in (X, τ).
The n^{th} Cesàro sections of a sequence $x = (x_k) \in X$ is given by

$$\sigma^n x = \frac{1}{n} \sum_{k=1}^{n} \sum_{j=1}^{k} x_j e^j.$$

We say that x has the property:
σK if $\|\sigma^n x - x\|_X \to 0, (n \to \infty)$,
$S\sigma K$ if $((|f(\sigma^n x) - f(x)|)_n) \in c_0, \forall f \in X^{'}$,
$F\sigma K$ if $(f(\sigma^n x)_n) \in c, \forall f \in X^{'}$,
σB if $(f(\sigma^n x)_n) \in \ell_\infty, \forall f \in X^{'}$, [3].

Now we are constructing a new definition:

$$r^{[n]} x = \frac{1}{Q_n} \sum_{k=1}^{n} q_k x_k e^k$$

is called the n^{th} Riesz section of x . This here r is the set $\{r^n : n \in \mathbb{N}\}$. In addition, an FK space is said to have rK space if $X \supset \phi$ and for each $x \in X$,

$$\frac{1}{Q_n} \sum_{k=1}^{n} q_k x^{(k)} \to x \quad , n \to \infty.$$

Every AK space is a rK space. For example ω, c_0 are both AK spaces and rK space [3], [14]. The transformation given by $q_n = \frac{q_1 s_1 + \cdots + q_n s_n}{Q_n}$ is called the Riesz mean (R, q_n) or simply the (R, q_n) mean, where (q_k) is a sequence of positive numbers and $Q_n = q_1 + \cdots + q_n$, [15].

In this paper, let $X = (x_{nk})$ be a matrix;

$$(x_{nk}) = \begin{cases} \frac{k.(q_k - q_{k+1})}{Q_n} & , \quad k < n \\ \frac{k.q_k}{Q_k} & , \quad k = n \\ 0 & , \quad k \geq n \end{cases}$$

and we suppose that $\sum_n x_{nk}$ convergent and

$$\sup_m \sum_k \left| \sum_{n=0}^{m} (x_{nk} - x_{nk-1}) \right| < \infty.$$

In line with this information rs containing σs can be seen. In other words if $x \in \sigma s$ then $x \in rs$.

Then

$$X^f = \left\{ \{f(\delta^k)\} : f \in X^{'} \right\}.$$

In addition

$$
\begin{aligned}
X^Y &= \{x : yx = (y_k x_k) \in Y \ \text{ for every } \ y \in X\} = (X \to Y), \\
X^\beta &= \{x : yx = (y_k x_k) \in cs \ \text{ for every } \ y \in X\} \\
&= \left\{ x : \sum_{k=1}^{\infty} x_k y_k \ \text{ exists for every } \ y \in X \right\}, \\
X^r &= \{x : yx = (y_k x_k) \in rs \ \text{ for every } \ y \in X\} \\
&= \left\{ x : \lim_n \frac{1}{Q_n} \sum_{k=1}^{n} \sum_{j=1}^{k} q_j x_j y_j \ \text{ exists for every } \ y \in X \right\}, \\
X^{rb} &= \left\{ x : \sup_n \frac{1}{Q_n} \left| \sum_{k=1}^{n} \sum_{j=1}^{k} q_j x_j y_j \right| < \infty \ \text{ for every } \ y \in X \right\}.
\end{aligned}
$$

Note that $\ell_\infty^\beta = c_0^\beta = c^\beta = \ell, \ell^\beta = \ell_\infty, cs^\beta = bv, bv^\beta = cs$, [16].

For example, it is claim $(rs)^r = \rho$:

$$
\begin{aligned}
(rs)^r &= ((cs)_R)^r \\
&= \{\alpha \in \omega : (\alpha x) \in (cs)_R, \forall x \in (cs)_R\} \\
&= \{\alpha \in \omega : A = R.diag(\alpha).R^{-1} \in (cs, cs)\} \\
&= \{\alpha \in \omega : A \in (cs, cs)\} \\
&= \{\alpha \in \omega : \sum_n a_{nk} \ \text{ convergent and } \sup_m \sum_k \left| \sum_{n=0}^{m} (a_{nk} - a_{nk-1}) \right| < \infty\} \\
&= \rho
\end{aligned}
$$

Let X, X_1 be sets of sequences. Then for $k = f, \beta, r, rb$
(a) $X \subset X^{kk}$, (b) $X^{kkk} = X^k$, (c) if $X \subset X_1$ *then* $X_1^k \subset X^k$ holds.

Theorem 1.1. *Let X be an FK space containing ϕ and $\lim_{n\to\infty} \frac{n}{Q_n} = 1$. Then*
(1)$X^\beta \subset X^r \subset X^{rb} \subset X^f$,
(2)If X is rK space then $X^f = X^r$,
(3)If X is an AD space then $X^r = X^{rb}$.

Proof. (2) Let $u \in X^r$ and define

$$f(x) = \lim_n \frac{1}{Q_n} \sum_{k=1}^{n} \sum_{j=1}^{k} q_j x_j u_j$$

for $x \in X$. Then $f \in X'$; by the Banach-Steinhaus [11, Theorem 1.0.4]. Also $f(e^p) = \lim_n \frac{1}{Q_n}(n - (p-1))q_p u_p = u_p, (p < n)$ so $u \in X^f$. Thus $X^r \subset X^f$. Now we show that $X^f \subset X^r$. Let $u \in X^f$. Since X is rK space

$$f(x) = \lim_n \frac{1}{Q_n} \sum_{k=1}^{n} \sum_{j=1}^{k} q_j x_j f(e^j) = \lim_n \frac{1}{Q_n} \sum_{k=1}^{n} \sum_{j=1}^{k} q_j x_j u_j,$$

for $x \in X$, then $u \in X^r$. Hence $X^f = X^r$.
(3) Let $u \in X^{rb}$ and define $f_n(x) = \frac{1}{Q_n}\sum_{k=1}^{n}\sum_{j=1}^{k} q_j x_j u_j$ for $x \in X$. Then $\{f_n\}$ is pointwise bounded, hence equicontinuous by [11, Theorem 7.0.2]. Since $\lim_n f(e^p) = u_p$ $(p < n)$ then $\phi \subset \{x : lim_n f_n(x) \ \ exists\}$. Hence $\{x : lim_n f_n(x) \ \ exists\}$ is closed subspace of X by the Convergence Lemma, [11, Theorem 1.0.5,7.0.3]. Since X is an AD space then $X = \{x : lim_n f_n(x) \ \ exists\} = \bar{\phi}$ and then $\lim_n f_n(x)$ exists for all $x \in X$. Thus $u \in X^r$. The opposite inclusion is trivial.
(1) $\bar{\phi} \subset X$ by the hypothesis. Since $\bar{\phi}$ is rK space, then $X^{rb} \subset (\bar{\phi})^{rb} = (\bar{\phi})^r = (\bar{\phi})^f = X^f$ by (2), (3) and [11, Theorem 7.2.4]. □

2 Main Results

In this section, we give the main results of this paper. We construct new important subspaces of a locally convex FK space X containing ϕ. Then we show that there is relation among these subspaces.

Definition 2.1. Let X be an FK space containing ϕ. Then following definitions hold.

$$W = W(X) = \{x \in X : x^{(k)} \to x \ \ \text{(weakly)} \ \ \text{in} \ \ X\}, [17],$$

$$\begin{aligned} RS &= RS(X) \\ &= \{x \in X : \frac{1}{Q_n}\sum_{k=1}^{n} q_k x^{(k)} \to x \ \ \text{in} \ \ X\} \\ &= \{x \in X : \lim_n \frac{1}{Q_n}\sum_{k=1}^{n}\sum_{j=1}^{k} q_j x_j e^j = x \ \ \} \\ &= \{x \in X : x \ \ \text{has} \ \ rK \ \ \text{in} \ \ X\}, \end{aligned}$$

$$\begin{aligned} RW &= RW(X) \\ &= \{x \in X : \frac{1}{Q_n}\sum_{k=1}^{n} q_k x^{(k)} \to x \ \ \text{(weakly)} \ \ \text{in} \ \ X\} \\ &= \{x \in X : \lim_n \frac{1}{Q_n}\sum_{k=1}^{n}\sum_{j=1}^{k} q_j x_j f(e^j) = f(x) \ \ \text{for all} \ \ f \in X'\} \\ &= \{x \in X : x \ \ \text{has} \ \ SrK \ \ \text{in} \ \ X\}, \end{aligned}$$

$$\begin{aligned} RF^+ &= RF^+(X) = rF^+(X) \\ &= \{x \in X : \left(\frac{1}{Q_n}\sum_{k=1}^{n} q_k x^{(k)}\right) \ \ \text{is weakly Cauchy in} \ \ X\} \\ &= \{x \in X : \{x_n f(e^n)\} \in rs \ \ \text{for all} \ \ f \in X'\} \\ &= \{x \in X : x \ \ \text{has} \ \ FrK \ \ \text{in} \ \ X\}, \end{aligned}$$

$$\begin{aligned} RB^+ &= RB^+(X) = rB^+(X) \\ &= \{x \in X : \left(\frac{1}{Q_n}\sum_{k=1}^{n} q_k x^{(k)}\right) \ \ \text{is bounded in} \ \ X\} \\ &= \{x \in X : \{x_n f(e^n)\} \in rb \ \ \text{for all} \ \ f \in X'\} \\ &= \{x \in X : x \ \ \text{has} \ \ rB \ \ \text{in} \ \ X\}. \end{aligned}$$

Also, $RF = RF^{+}(X) \cap X$ and $RB = RB^{+}(X) \cap X$.

Definition 2.2. Sequence sets of above definitions show that:

1. $X_{rK} = RS = RS(X) = \{x \in X : x \text{ has } rK\} \subset X$
2. $X_{SrK} = RW = RW(X) = \{x \in X : x \text{ has } SrK\} \subset X$
3. $X_{FrK} = RF = RF(X) = \{x \in X : x \text{ has } FrK\} \subset X$
4. $X_{rB} = RB = RB(X) = \{x \in X : x \text{ has } rB\} \subset X$

Corollary 2.1. *By definition 2.1 we obtain from following results:*

1. X *has* FrK *iff* $X \subset RF$ *,i.e.,* $X = RF$,

2. X *has* rB *iff* $X \subset RB$ *,i.e.,* $X = RB$.

We introduce some inclusions which are similar to given in [11]. Then we shall study above properties in accordance with previous investigations on related sectional properties such as [8, 11, 13].

Theorem 2.2. *Let* X *be an* FK *space containing* ϕ*. Then*

$$\phi \subset RS \subset RW \subset RF \subset RB \subset X$$

and

$$\phi \subset RS \subset RW \subset \bar{\phi}.$$

Proof.
First conclusion is obvious by Definition 2.1. We show that the inclusion $RW \subset \bar{\phi}$. Let $f \in X'$ and $f = 0$ on ϕ. The definition of RW shows that $f = 0$ on RW. Thus, the Hanh-Banach theorem gives the result.

Theorem 2.3. *The subspaces* $E = RS, RW, RF, RB, RF^{+}$ *and* RB^{+} *of* X, FK *space are monotone, i. e., if* $X \subset Y$ *then* $E(X) \subset E(Y)$.

Proof.
Let $E = RS$, $X \subset Y$ and $x \in RS(X)$. Then by

$$\lim_{n} \frac{1}{Q_n} \sum_{k=1}^{n} \sum_{j=1}^{k} q_j x_j e^j = x \in X$$

and $X \subset Y$, we give that

$$\lim_{n} \frac{1}{Q_n} \sum_{k=1}^{n} \sum_{j=1}^{k} q_j x_j e^j = x \in Y.$$

Hence $RS(X) \subset RS(Y)$ conclusion is ensure. The others can be proved in a similar way.

Theorem 2.4. *Let* X *be an* FK *space containing* ϕ*. Then* $RB^{+} = (X^f)^{rb}$.

Proof.
By Definition 2.1, $z \in RB^{+}$ if and only if $zu \in rb$ for each $u \in X^f$. Hence $RB^{+} \subset (X^f)^{rb}$ holds. The converse inclusion is trivial. This is precisely the claim.

Theorem 2.5. *Let* X *be an* FK *space containing* ϕ*. Then* RB^{+} *is the same for all* FK *spaces* Y *between* $\bar{\phi}$ *and* X*; i. e. ,*$\bar{\phi} \subset Y \subset X$ *implies* $RB^{+}(Y) = RB^{+}(X)$*. Here the closure of* ϕ *is calculated in* X*.*

Proof.
By Theorem 2.3, we have $RB^+(\bar{\phi}) \subset RB^+(Y) \subset RB^+(X)$. By Theorem 2.4, the first and last are equal.

Theorem 2.6. *Let X be an FK space such that $RB \supset \bar{\phi}$. Then $\bar{\phi}$ has rK and $RS = RW = \bar{\phi}$.*

Proof.
By Theorem 2.2, $\phi \subset RS \subset RW \subset \bar{\phi} \subset RB$. Firstly, suppose that X has RB. Define $f_n : X \to X$ by $x \to f_n(x) = x - \frac{1}{Q_n}\sum_{k=1}^{n} q_k x^{(k)}$. Then $\{f_n\}$ is pointwise bounded, hence equicontinuous by [11]. Since $f_n \to 0$ on ϕ then also $f_n \to 0$ on $\bar{\phi}$ by [11]. As a result, the proof is complete.

Theorem 2.7. *Let X be an FK space containing ϕ. Then $RF^+ = (X^f)^r$.*

Proof.
This can be proved as in Theorem 2.4, with rs instead of rb.

Theorem 2.8. *Let X be an FK space containing ϕ. Then RF^+ is the same for all FK spaces Y between $\bar{\phi}$ and X; i. e. ,$\bar{\phi} \subset Y \subset X$ implies $RF^+(Y) = RF^+(X)$. Here the closure of ϕ is calculated in X.*

Proof.
The proof is similar to that of Theorem 2.5.

Theorem 2.9. *Let X be an FK space in which $\bar{\phi}$ has rK. Then $RF^+ = (\bar{\phi})^{rr}$.*

Proof.
It is obvious that $RF^+ = (X^f)^r$ by Theorem 2.7. Since $X^f = (\bar{\phi})^f$ by [11], we have $(X^f)^r = (\bar{\phi})^{fr}$. Thus by Theorem 1.1 the result follows.

Theorem 2.10. *Let X be an FK space containing ϕ. Then X has FrK if and only if $\bar{\phi}$ has rK and $X \subset (\bar{\phi})^{rr}$.*

Proof.
Necessity. X has rB since $RF \subset RB$, so $\bar{\phi}$ has rK. If $\bar{\phi}$ has rK then $X \subset RF^+ = (\bar{\phi})^{rr}$. Hence $X \subset (\bar{\phi})^{rr}$. Sufficiency is given by Theorem 2.9.

Theorem 2.11. *Let X be an FK space containing ϕ. The following statements are equivalent:*
(1) X has FrK (or RF),
(2) $X \subset (RS)^{rr}$,
(3) $X \subset (RW)^{rr}$,
(4) $X = (RF)^{rr}$,
(5) $X^r = (RS)^r$,
(6) $X^r = (RF)^r$.

Proof.
Since $RS \subset RW \subset RF \subset X$, it is trivial that (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4). If (4) is true, then

$$X^f \subset (RF)^r = (X^{fr})^r \subset X^r$$

so (1) is true by Theorem 1.1. If (1) holds, then Theorem 2.10 implies that $\bar{\phi} = RS$ which means (2) holds. The equivalence of (5),(6) with others is clear.

Theorem 2.12. *Let X be an FK space containing ϕ. The following are equivalent:*
(1) X has SrK ,
(2) X has rK ,
(3) $X^r = X'$.

Proof.
By Theorem 2.2, it is clear (2) implies (1). Conversely if X has SrK it must have AD from $RW \subset \bar{\phi}$ by Theorem 2.2. It also has rB since $RW \subset RB$. Thus X has rK by Theorem 2.6, this proves that (1) and (2) are equivalent. Assume that (3) holds. Let $f \in X'$, then there exists $u \in X^r$ such that

$$f(x) = \lim_n \frac{1}{Q_n} \sum_{k=1}^{n} \sum_{j=1}^{k} u_j x_j q_j$$

for $x \in X$. Since $u_j = f(e^j)$, it follows that each $x \in RW$ which shows that (3) implies (1). Let X has rK, then by Theorem 2.2 it has SrK. So, by Definition 2.1, for all $f \in X'$ there is

$$f(x) = \lim_n \frac{1}{Q_n} \sum_{k=1}^{n} \sum_{j=1}^{k} u_j x_j q_j$$

such that $u \in X^r$ which is $u_j = f(e^j), (\forall x \in X)$. This shows that (2) implies (3).

Theorem 2.13. *Let X be an FK space containing ϕ. The following are equivalent:*
(1) RW is closed in X,
(2) $\bar{\phi} \subset RB$,
(3) $\bar{\phi} \subset RF$,
(4) $\bar{\phi} = RW$,
(5) $\bar{\phi} = RS$,
(6) RS is closed in X.

Proof.
(2) $\Rightarrow$ (5): By Theorem 2.6, $\bar{\phi}$ has rK, i.e., $\bar{\phi} \subset RS$. The opposite inclusion is Theorem 2.2. Note that (5) implies (4), (5) implies (3) and (3) implies (2) because of Theorem 2.2;

$$RS \subset RW \subset \bar{\phi}, RW \subset RF \subset RB;$$

(1)$\Rightarrow$ (4) and (6)$\Rightarrow$ (5) since $\phi \subset RS \subset RW \subset \bar{\phi}$. Finally (4)$\Rightarrow$ (1) and (5)$\Rightarrow$ (6).

Theorem 2.14. *Let X be an FK space containing ϕ. Then X has rB property if and only if $X^f \subset X^{rb}$.*

Proof.
Necessity Let X be rB property. Then $X \subset RB^+ = (X^f)^{rb}$ and $X^{rb} \supset (X^f)^{rbrb} \supset X^f$. Sufficiency is clear.

Theorem 2.15. *Let X be an FK space containing ϕ. Then X has rF^+ property if and only if $X^f \subset X^r$.*

Proof.
The proof is similar to that of Theorem 2.14.

3 Conclusion

In this study, we determined a new $r-$ and $rb-$ type duality of a sequence space X containing ϕ. Moreover, we developed some new subspaces which are the importance of each one on topological sequence spaces theory. We study the subspaces RS, RW, RF^+ and RB^+ for a locally convex FK space X containing ϕ, the space of finite sequences. Then, we showed that there is relation among these subspaces. Furthermore, we examined monotone of the distinguished subspaces. Finally, we proved some theorems related to the $f-$, $r-$ and $rb-$ duality of a sequence spaces X.

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Competing Interests

Authors have declared that no competing interests exist.

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