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Cusp Forms Whose Fourier Coefficients Involve Dirichlet Series

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Author's contribution

The sole author designed, analyzed and interpreted and prepared the manuscript.

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Abstract

We present some cusp forms on the full modular group Γ_1, using the properties of eigenfunctions, nonanalytic Poincare series and Hecke operators T_n. Further, the Fourier coefficients of cusp forms $T_n f$ on Γ_1 are given in terms of Dirichlet series associated to the Fourier coefficients of cusp form f of weight k.

Keywords: Cusp forms; eigenfunctions; nonanalytic Poincare series; Dirichlet series; Hecke operators.

2010 mathematics subject classification: 11F12, 11F03, 11F41, 11F30.

1 Introduction

Let k be a positive integer and denote by S_k the space of cusp forms and by M_k the space of modular forms of weight k on the full modular group Γ_1. We shall use H to denote the upper half plane, $\mathbb{C}$ for the set of complex numbers.

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Let $f, f' \in M_k$ such that f or f' is a cusp form. The Petersson scalar product is defined by

$$\langle f, f' \rangle = \int_K f(\tau)\overline{f'(\tau)} y^k dV$$

in [1]. Where $\tau = x + iy$, $dV = \frac{dxdy}{y^2}$ and K is a fundamental domain for the action of Γ_1 on H.

In [2, p. 115], nonanalytic Poincare series is defined by

$$G_\nu(\tau \mid z) = \sum_{(c,d)=1} \sum \frac{\exp\{2\pi i(\nu+\kappa)M\tau\}}{\upsilon(M)(c\tau+d)^k \left|c\tau+d\right|^z} \tag{1}$$

where $M = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma_1$, $| c\tau + d |^z$ is the Hecke convergence factor, Im τ>0, ν is an arbitrary integer and υ is a multiplier system (MS) for Γ_1 in the weight k. The number κ is determined from υ by $\upsilon(S) = e^{2\pi i \kappa}, 0 \le \kappa < 1, S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Eventually z can be thought of as an arbitrary complex number, but in order to guarantee absolute convergence of the double series (1) we assume initially that Rez>2-k. Uniform convergence of the series of absolute values implies that $G_\nu(\tau \mid z)$ is holomorphic (in the variable z) in the half-plane Rez>2-k and, as a function of $\tau \in$ H, it satisfies the transformation formula,

$$G_\nu(M\tau \mid z) = \upsilon(M)(c\tau+d)^k \mid c\tau+d \mid^z G_\nu(\tau \mid z) \tag{2}$$

for all $M = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma_1$.

In [2, p.118], the Fourier expansion of $G_\nu(\tau \mid z)$ is given by

$$G_\nu(\tau \mid z) - 2e^{2\pi i(\nu+\kappa)\tau} = 2i^{-k} \frac{(2\pi)^{k+z}}{\Gamma(z/2)} \sum_{n=0}^{\infty} (n+\kappa)^{k+z-1} e^{2\pi i(n+\kappa)\tau} \sum_{p=0}^{\infty} \frac{\left\{-4\pi^2(n+\kappa)(\nu+\kappa)\right\}^p}{p!\Gamma(k+p+z/2)}$$
$$\times \sigma(4\pi(n+\kappa)y, k+p+z/2, z/2) Z_n(z/2+k/2+p)$$

$$+2i^{-k} \frac{(2\pi)^{k+z}}{\Gamma(z/2)} \sum_{n=0}^{\infty} (n-\kappa)^{k+z-1} e^{-2\pi i(n-\kappa)\bar{\tau}} \sum_{p=0}^{\infty} \frac{\left\{-4\pi^2(n-\kappa)(\nu+\kappa)\right\}^p}{p!\Gamma(k+p+z/2)}$$
$$\times \sigma(4\pi(n-\kappa)y, z/2, k+p+z/2) Z_{-n}(z/2+k/2+p) \tag{3}$$

where $Z_n(w)=\sum_{c=1}^{\infty} A_{c,\nu}(n,\upsilon)c^{-2w}$ is Selberg's Kloosterman zeta-function and $\sigma(\eta,\alpha,\beta)=\int_0^{\infty}(u+1)^{\alpha-1}u^{\beta-1}e^{-\eta u}du$ is the notation of Siegel.

In [2, p.125], the function $F_{\nu}(\tau \mid z)$ is defined by

$$F_{\nu}(\tau \mid z)=y^{z/2}G_{\nu}(\tau \mid z) \tag{4}$$

as a function of τ and z. Where $\tau = x+iy$. It follows from (2) that, $F_{\nu}(\tau \mid z)$ satisfies the transformation formulae,

$$F_{\nu}(M\tau \mid z)=\upsilon(M)(c\tau+d)^k F_{\nu}(\tau \mid z)$$

By the Fourier expansion (3) of $G_{\nu}(\tau \mid z)$ and from (4), we obtain the Fourier expansion of $F_{\nu}(\tau \mid z)$ at the cusp point ∞ of the form

$$F_{\nu}(\tau \mid z)=\sum_{n=0}^{\infty} a_1(n)e^{2\pi i(n+\kappa)\tau}+\sum_{n=0}^{\infty} a_2(n)e^{-2\pi i(n-\kappa)\bar{\tau}}$$

where the Fourier coefficients $a_1(n)$ and $a_2(n)$ depend upon z. Hence, $F_{\nu}(\tau \mid z)$ is a modular form of weight k and MS υ.

In [2, p. 125], the following lemma is given.

Lemma 1.1. Suppose $\nu+\kappa>0$, Rez>2-k and f(τ) is a cusp form of weight k and MS υ on Γ_1. Then,

$$\langle F_{\nu},f\rangle = 2\bar{b}_{\nu}\Gamma(k-1+z/2)\{4\pi(\nu+\kappa)\}^{1-k-z/2}$$

where $f(\tau)=\sum_{n+\kappa>0} b_n e^{2\pi i(n+\kappa)\tau}$ and the bar denotes the conjugate complex number.

For $n=\nu+\kappa$, we shall write $F_{k-t,n}(\tau \mid z)$ instead of $F_{\nu}(\tau \mid z)$. Thus, we have

$$F_{k-t,n}(\tau \mid z)=y^{z/2}G_{k-t,n}(\tau \mid z)=y^{z/2}\sum_{(c,d)=1}\sum \frac{\exp\{(2\pi in)M\tau\}}{\upsilon(M)(c\tau+d)^{k-t}|c\tau+d|^{z}}$$

In [3], the Hecke operator T_n is defined on M_k by the equation

$$(T_n f)(\tau)=n^{k-1}\sum_{d|n} d^{-k}\sum_{b=0}^{d-1} f\left(\frac{n\tau+bd}{d^2}\right)$$

for a fixed integer k and any n=1,2,…

Theorem 1.2. If $f \in M_k$ and has the Fourier expansion

$$f(\tau) = \sum_{m=0}^{\infty} a(m) e^{2\pi i m\tau}$$

then $T_n f$ has the Fourier expansion

$$(T_n f)(\tau) = \sum_{m=0}^{\infty} \gamma_n(m) e^{2\pi i m\tau}$$

where,

$$\gamma_n(m) = \sum_{d|(n,m)} d^{k-1} a(\frac{nm}{d^2}). \quad (5)$$

for n=1,2,…

A nonzero function f satisfying a relation of the form $T_n f = \lambda(n) f$ for some complex scalar $\lambda(n)$ is called an eigenfunction (eigenform) of the operator T_n, and the scalar $\lambda(n)$ is called an eigenvalue of T_n. If f is an eigenfunction for every Hecke operators T_n, $n \geq 1$, then f is called a simultaneous eigenfunction. Since $\dim S_k = 1$, for $k = 12,16,18,20,22$ and 26, each T_n has eigenfunction in S_k for each of these values of k. The respective cusp forms $\Delta, \Delta G_4, \Delta G_6, \Delta G_8, \Delta G_{10}$ and ΔG_{14} are eigenfunctions for each T_n. Where $\Delta(\tau) = g_2^3(\tau) - 27 g_3^2(\tau)$, $g_2(\tau) = 60 G_4(\tau)$, $g_3(\tau) = 140 G_6(\tau)$ and $G_k(\tau) = \sum_{(m,n)\neq(0,0)} (m + n\tau)^{-k}$ [3].

Hecke found a remarkable connection between each modular form with Fourier series

$$f(\tau) = a(0) + \sum_{n=1}^{\infty} a(n) e^{2\pi i n\tau}$$

and the Dirichlet series

$$L_f(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} \quad (6)$$

formed with the same coefficients (except for a(0)). If $f \in M_{2k}$ then $a(n) = O(n^k)$ if f is a cusp form, and $a(n) = O(n^{2k-1})$ if f is not a cusp form. Therefore, the Dirichlet series in (6) converges absolutely for $\sigma = Re(s) > k + 1$ if f is a cusp form, and for $\sigma > 2k$ if f is not a cusp form. For cusp forms, it has been shown that $a(n) = O\left(n^{k-\frac{1}{4}+\epsilon}\right)$ for every $\epsilon > 0$ [3].

In [3], the following theorem is given.

Theorem 1.3. If f is modular and bounded in H then f is constant.

In [4], W. Kohnen proved the following theorem using analytic Poincare series and the properties of inner product.

Theorem 1.4. The function

$$W_g(f)(z) = \sum_{n \geq 1} n^{k-t-1} L'_{f,g;n}(k-1) e^{2\pi i n z}, \ (z \in \mathrm{H})$$

is a cusp form of weight $k-t$ on Γ_1. In fact, the map $W_g : S_k \to S_{k-t}, f \mapsto c_{k,t} W_g(f)$ is the adjoint w.r.t. the usual Petersson scalar products of the map $S_{k-t} \to S_k, h \mapsto gh.$ Where

$$c_{k,t} := \frac{\Gamma(k-1)}{\Gamma(k-t-1)(4\pi)^t}, L'_{f,g;n}(s) = \sum_{m \geq 1} \frac{a(m+n)\overline{b(m)}}{(m+n)^s} \quad (7)$$

In [5], Min Ho Lee obtained the Fourier coefficients of Siegel cusp form $\phi_g^* f$ in terms of Dirichlet series of Rankin type associated to the Fourier coefficients of Siegel cusp forms f and g.

In [6], author proved the following theorems, using the nonanalytic Ponicare series and the properties of inner product.

Theorem 1.5. Let k be an integer with k>2. Let $g_1(\tau)$ be a modular function with respect to Γ_1 which is analytic on H and $f(\tau) \in S_k$. Then the function, for $\tau \in H$ and $Rez > 2-k$,

$$U_{g_1}(f)(\tau) = \sum_{n \geq 1} a(T_p f, n) e^{2\pi i n \tau}$$

is a cusp form of weight k on Γ_1. Where

$$a(T_p f, n) = \frac{\overline{\Gamma(k-1)(4\pi)^{\frac{z}{2}}}}{2\Gamma(k-1+\frac{z}{2})} \overline{n^{k-1+\frac{z}{2}}} L_{f,g_1;n}(k-1)$$

for $n = p$ and $m, p = 1,2,...$ and $L_{f,g_1;n}(s) = \sum_{m \geq 1} \frac{\overline{a(m+n)} b(m)}{(m+n)^s}$.

Theorem 1.6. Let k be an integer with k>2. Let $g_1(\tau)$ be a modular function with respect to Γ_1 which is analytic on H and $f(\tau) \in S_k$. Then the function

$$W_{g_1}(f)(\tau) = \sum_{n \geq 1} a(T_p f, n) e^{2\pi i n \tau}$$

is a cusp form of weight k on Γ_1. Where

$$a(T_p f, n) = n^{k-1} L'_{f,g_1;n}(k-1)$$

for $n = p$ and $m, p = 1,2,\ldots$ and $L'_{f,g_1,n}(k-1)$ is given by (7).

This paper is a continuation of previous work [6]. In this paper, some cusp forms of integer weight on Γ_1 are obtained , using nonanalytic Poincare series and the properties of eigenfunction. Further, we will write the Fourier coefficients of cusp forms $T_n f$ on Γ_1 in terms of Dirichlet series associated to the Fourier coefficients of cusp form f of weight k. Here, we follow the method of W. Kohnen [4], who obtained a similar result using analytic Poincare series.

For several recent results concerning Modular forms, we refer the reader to [7-18].

2 The Main Results

Firstly, we start by the following theorem:

Theorem 2.1. Let k be an integer with k>2 and $f(\tau) \in S_k$. If $f(\tau)$ is an eigenfunction for all T_n, then the function, for $\tau \in$ H and $Rez > 2-k$,

$$U(f)(\tau) = \sum_{n \geq 1} a(T_p f, n) e^{2\pi i n \tau}$$

is a cusp form of weight k on Γ_1. Where

$$a(T_p f, n) = \frac{\lambda(n)\Gamma(k-1)\overline{(4\pi)^{\frac{z}{2}}}}{\overline{2\Gamma(k-1+\frac{z}{2})}} n^{\overline{k-1+\frac{z}{2}}} L_f(k-1) \tag{8}$$

for $n = p$ and $m, p = 1,2,\ldots$, $L_f(k-1)$ is the Dirichlet series and $\lambda(n)$ is an eigenvalue of T_n.

Proof. Let $f(\tau) \in S_k$ and k an integer with k>2. Let $T_p : S_k \to S_k$ be a Hecke operator. Since $f(\tau)$ is an eigenfunction for all T_n, using Lemma 1.1 and from Petersson scalar product, we obtain

$$\begin{aligned} w_0 2.\overline{a(T_p f, n)} &= \langle T_p f, F_{k,n} \rangle \\ &= \langle \lambda(n) f, F_{k,n} \rangle \\ &= \lambda(n) \langle f, F_{k,n} \rangle \\ &= \lambda(n) \int_K f(\tau) \overline{F_{k,n}}(\tau) y^k dV \end{aligned}$$

where $w_0 = \frac{\Gamma(k-1+\frac{z}{2})}{(4\pi n)^{k-1+\frac{z}{2}}}$. Hence, we get

$$2a(T_p f, n) = \frac{\lambda(n)}{\overline{w_0}} \int_0^{\infty} a(\bar{f}, n, y) e^{-2\pi n y} y^{k-2} dy$$

where $a(\bar{f}, n, y)$ is the nth Fourier coefficient of $\bar{f}$ w.r.t. the variable $e^{2\pi i x}$. Using the Fourier expansion of f, we obtain

$$a(\bar{f}, n, y) = \sum_{n \geq 1} a(n) e^{-2\pi n y}$$

By Mellin's transform, we find,

$$a(T_p f, n) = \frac{\lambda(n)\Gamma(k-1)\overline{(4\pi)^{\frac{z}{2}}}}{\overline{2\Gamma(k-1+\frac{z}{2})}} \overline{n^{k-1+\frac{z}{2}}} \sum_{n \geq 1} \frac{a(n)}{n^{k-1}}$$

and by (5)

$$\gamma_n(m) = a(T_p f, n) = \frac{\lambda(n)\Gamma(k-1)\overline{(4\pi)^{\frac{z}{2}}}}{\overline{2\Gamma(k-1+\frac{z}{2})}} \overline{n^{k-1+\frac{z}{2}}} L_f(k-1)$$

for $n = p$ and $m, p = 1, 2, \ldots$ Where $\Gamma(s) = \int_0^{\infty} t^{s-1} e^{-t} dt,\ Re(s) > 0$. This completes the proof.

The proof of the following Theorem is similar to that of Theorem 2.1 and use Theorem 1.4.

Theorem 2.2. Let k be an integer with k>2 and $f(\tau) \in S_k$. If $f(\tau)$ is an eigenfunction for all T_n, then the function

$$W(f)(\tau) = \sum_{n \geq 1} a(T_p f, n) e^{2\pi i n \tau}$$

is a cusp form of weight k on Γ_1. Where

$$a(T_p f, n) = \lambda(n) n^{k-1} L_f(k-1) \tag{9}$$

for $n = p$ and $m, p = 1, 2, \ldots$

Now suppose that g_1 is a modular function with respect to Γ_1 which is bounded in H. Then, by Theorem 1.3, $g_1(\tau)$ is constant. Let $g_1(\tau) = 1$. Thus, we can give the following corollaries which are special cases of Theorems 1.5 and 1.6, respectively. Their proofs are similar to that of Theorem 2.1.

Corollary 2.1. Let k be an integer with k>2. Let $g_1(\tau)$ be a modular function with respect to Γ_1 which is bounded in H and $f(\tau) \in S_k$. Then the function, for $\tau \in H$ and $Re z > 2 - k$,

$$U_1(f)(\tau) = \sum_{n\geq 1} a(T_p f, n) e^{2\pi i n \tau}$$

is a cusp form of weight k on Γ_1. Where

$$a(T_p f, n) = \frac{\Gamma(k-1)\overline{(4\pi)^{\frac{z}{2}}}}{2\Gamma(k-1+\frac{z}{2})} n^{\overline{k-1+\frac{z}{2}}} L_f(k-1) \tag{10}$$

for $n = p$ and $m, p = 1,2,...$

Corollary 2.2. Let k be an integer with k>2. Let $g_1(\tau)$ be a modular function with respect to Γ_1 which is bounded in H and $f(\tau) \in S_k$. Then the function

$$W_1(f)(\tau) = \sum_{n\geq 1} a(T_p f, n) e^{2\pi i n \tau}$$

is a cusp form of weight k on Γ_1. Where

$$a(T_p f, n) = n^{k-1} L_f(k-1) \tag{11}$$

for $n = p$ and $m, p = 1,2,...$

3 Numerical Examples

Example 1. Let $f(\tau) = \Delta(\tau) G_{14}(\tau)$. Since the function $\Delta(\tau)G_{14}(\tau)$ is a cusp form of weight 26 and from (8) and (9), we have

$$\gamma_n(m) = a(T_p \Delta G_{14}, n) = \frac{25!\lambda(n)\overline{(4\pi)^{\frac{z}{2}}}}{2\Gamma(25+\frac{z}{2})} n^{\overline{25+\frac{z}{2}}} L_f(25) .$$

and

$$\gamma_n(m) = a(T_p \Delta G_{14}, n) = \lambda(n) n^{25} L_f(25)$$

as the Fourier coefficients of $T_n \Delta G_{14}$, respectively.

Example 2. Let $f(\tau) = \Delta(\tau)$. Since the discriminant function $\Delta(\tau)$ is a cusp form of weight 12 and by (10) and (11), we get

$$\gamma_n(m) = a(T_p\Delta, n) = \frac{11!\overline{(4\pi)^{\frac{z}{2}}}}{2\Gamma(11+\frac{z}{2})}\overline{n^{11+\frac{z}{2}}}L_f(11)$$

and

$$\gamma_n(m) = a(T_p\Delta, n) = n^{11}L_f(11)$$

as the Fourier coefficients of $T_n\Delta$, respectively.

4 Conclusion

Some cusp forms on the full modular group are obtained, using the properties of nonanalytic Poincare series, eigenfunctions and inner product. Further, the Fourier coefficients of cusp forms $T_n f$ on Γ_1 are given in terms of Dirichlet series associated to the Fourier coefficients of cusp form f of weight k. For example, one of these Fourier coefficients is $a(T_p f, n) = n^{k-1}L_f(k-1)$.

The open problem: What are the applications of these Fourier coefficients in Representation Theory ?

Competıng Interests

Author has declared that no competing interests exist.

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