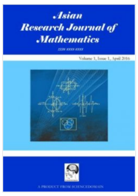

Asian Research Journal of Mathematics
2(3): 1-11, 2017; Article no.ARJOM.30465
ISSN: 2456-477X

SCIENCEDOMAIN *international*
www.sciencedomain.org

Convergence and Stability of Split-Step Milstein Schemes for Stochastic Differential Equations

Lingzhi Teng[1*], Haomin Zhang[1] and Xiaoting Tao[1]

[1] *College of Science, Guilin University of Technology, Guilin 541004, Guangxi, P.R. China.*

Authors' contributions

This work was carried out in collaboration between all authors. Author LT designed the study, performed the statistical analysis, wrote the protocol and wrote the first draft of the manuscript. Author HZ managed the analyses of the study. Author XT managed literature searches. All authors read and approved the final manuscript.

Article Information

DOI: 10.9734/ARJOM/2017/30465
Editor(s):
(1) Palle E. T. Jorgensen, Department of Mathematics, The University of Iowa, Iowa City, Iowa, USA.
Reviewers:
(1) Andriy Yurachkivsky, Taras Shevchenko National University of Kyiv, Ukraine.
(2) Rabha W. Ibrahim, University of Malaya, Malaysia.
(3) Shahab Ud-Din Khan, COMSATS Institute of Information Technology, Pakistan.
(4) Hakeem Ullah, Abdul Wali Khan University, Pakistan.
(5) Hui Li, Icahn medicine school at Mount Sinai, Egypt.
Complete Peer review History: http://www.sciencedomain.org/review-history/17638

Received: 10th November 2016
Accepted: 13th January 2017
Published: 27th January 2017

Original Research Article

Abstract

In this paper, the mean square convergence and stability of the split-step theta-Milstein schemes for stochastic differential equations are discussed. First, it is shown that these methods are mean square convergent with strong order 1. Then, we investigate the mean square stability of the split-step theta-Milstein methods. Finally, numerical examples are presented to illustrate the theoretical results.

Keywords: Stochastic differential equations; Mean square convergence; Mean square stability; Split-step theta-Milstein schemes.

Mathematics Subject Classification(2010): 60H10, 60H35, 65C30.

**Corresponding author: E-mail: 1391193656@qq.com;*

1 Introduction

Stochastic differential equations (SDEs) are widely utilized for mathematical models in many ways such as finance, biology, mechanics, ecology, chemistry and so on [1, 2, 3, 4, 5, 6, 7]. Due to most of the SDEs cannot be solved explicitly, numerical approximations became to be an important tool for studying them. Here we consider numerical integration of stochastic differential equations (SDEs) in the Itô sense of the following form:

$$\mathrm{d}x(t) = f(x(t))\mathrm{d}t + g(x(t))\mathrm{d}w(t), \qquad t \in [0, T], \tag{1.1}$$

with initial data $x(0) = x_0 \in \mathbb{R}^d$ $w(t)$ is a one-dimensional standard Wiener process and the functions $f, g : \mathbb{R}^d \mapsto \mathbb{R}^d$. Due to advantages of the split-step method in flexibility and stability of dealing with stiffness, we introduce the split-step theta-Milstein (SSTM) approximation

$$\begin{cases} y_n = x_n + \theta f(y_n)\Delta + (1-\theta) f(x_n)\Delta, \\ x_{n+1} = y_n + g(y_n)\Delta w_n + \frac{1}{2} L^1 g(y_n)(|\Delta w_n|^2 - \Delta), \end{cases} \tag{1.2}$$

where the parameter $\theta \in [0, 1]$, $\Delta = T/N$ is a stepsize, $N \in \mathbb{N}$, for a scalar differential function φ, $L^1\varphi = \nabla\varphi g$ and $\Delta w_n = w(t_{n+1}) - w(t_n)$ denotes the increment of Brownian motion.

In recent years, some Milstein schemes have been widely used to solve stochastic differential equations. The explicit Milstein method [8] is strongly convergent with order one, but in the mean-square sense, the explicit Milstein scheme generally does not converge to the exact solution of the SDEs with superlinearly growing drift coefficient. So Wang and Gan [9] introduced a tamed version of the Milstein scheme for SDEs with commutative noise, and in that paper, they obtained strongly convergent with order one in the non-globally Lipschitz conditions on the diffusion coefficient. In [10], Higham, Mao and Szpruch proposed a new Milstein type scheme for simulating stochastic differential equations (SDEs) with highly nonlinear coefficients, and analyzed multi-level Monte Carlo simulations for mean-reverting financial models with polynomial growth in the diffusion term. Nevertheless, the mean square stability analysis of Milstein methods is focused on linear SDEs [10, 11, 12], and there exists a bit number of results on stability analysis for nonlinear SDEs.

In this paper, the convergence and mean-square stability of the SSTM-method for nonlinear SDEs are studied. The organization of the paper is the following. In the next section, uniform boundedness of pth moments is obtained. We also introduce some notations and assumption of Eq.(1) in this section. The strong convergence order of SSTM-scheme is established in Section 3. In Section 4, the mean square stability of scheme (1.2) is presented. Finally, several interesting examples will be given to illustrate the theory.

2 Uniform Boundedness of the pth Moment

Throughout this paper, we use the following notations. Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$, which satisfies the usual conditions (increasing and right continuous, $\mathcal{F}_0$ contains all P-null sets). The Brownian motion $w(t)$ is defined on $(\Omega, \mathcal{F}, P)$. We let $|\cdot|$ denote both the Euclidean norm in R^n. If x is a vector, its transpose is denoted by x^T and the inner product is denoted by $\langle x, y\rangle = x^T y$ for $x, y \in \mathbb{R}^d$. We use C to denote a generic positive constant independent of the stepsize Δ and may vary from place to place.

We make following assumption to ensure the existence and uniqueness of the global solution of the SDEs (1.1).

Assumption 1. *There exist constants K, μ, δ such that for any $x, y \in \mathbb{R}^d$*

$$\begin{array}{l} |f(x) - f(y)|^2 \leq K|x - y|^2, \\ |g(x) - g(y)|^2 \leq \mu|x - y|^2, \\ |f(x)|^2 \vee |g(x)|^2 \leq K^{'}(1 + |x|^2), \\ |L^1 g(x) - L^1 g(y)|^2 \leq \delta|x - y|^2. \end{array} \tag{2.1}$$

Remark 2.1. *From Assumption 1, one easily deduce that*

$$|x^T f(x)| \vee |f(x)|^2 \vee |g(x)|^2 \vee |L^1 g(x)|^2 \leq \alpha + \beta|x|^2, \qquad x \in \mathbb{R}^d, \tag{2.2}$$

where

$$\alpha := 2|f(0)|^2 \vee 2|g(0)|^2 \vee 2|L^1 g(0)|^2 \quad \text{and} \quad \beta := (K + \frac{1}{2}) \vee 2K \vee 2\mu \vee 2\delta. \tag{2.3}$$

Definition 2.1. (See [13].)*The solution $x(t) \equiv 0$ of the system (1.1) in $\mathbb{R}^d$ is said to be p-stable ($p > 0$), if*

$$\sup_{t \geq 0} \mathbb{E}|x(t)|^p \to 0. \tag{2.4}$$

When $p = 2$, it is usually said to be mean-square stable.

Definition 2.2. *Assume that the conditions (2.1)C(2.4) hold. For a given stepsize Δ, a numerical method is said to be mean-square stable if for initial data x_0 the numerical solution x_n produced by the method satisfies*

$$\lim_{n \to \infty} \mathbb{E}[x_n^2] = 0. \tag{2.5}$$

Lemma 2.1. *Suppose that $f : \mathbb{R}^d \mapsto \mathbb{R}^d$ satisfies (2.2) and (2.3). If $\Delta < \min\{1, 1/(4\beta)\}$, then the x_n and y_n produced by (1.2) satisfy*

$$\mathbb{E}[y_n^2] \leq 5\mathbb{E}[x_n^2] + C, \tag{2.6}$$

where $0 \leq n \leq N$ and C is a positive constant.

Proof. It follows from (1.2) that

$$y_n = x_n + (\theta f(y_n) + (1 - \theta) f(x_n))\Delta. \tag{2.7}$$

Squaring both sides of (2.7), we obtain

$$\begin{array}{l} |y_n|^2 = |x_n|^2 + \theta^2\Delta^2|f(y_n)|^2 + (1 - \theta)^2\Delta^2|f(x_n)|^2 + 2\theta\Delta x_n^T f(y_n) \\ \quad +2(1 - \theta)\Delta x_n^T f(x_n) + 2\theta(1 - \theta)\Delta^2\langle f(y_n), f(x_n)\rangle. \end{array} \tag{2.8}$$

Using the inequality $x^T y \leq |x|^2 + |y|^2$, then

$$\begin{array}{rl} |y_n|^2 \leq & |x_n|^2 + \theta^2\Delta^2|f(y_n)|^2 + (1 - \theta)^2\Delta^2|f(x_n)|^2 + \theta\Delta(|x_n|^2 + |f(y_n)|^2) \\ & +(1 - \theta)\Delta(|x_n|^2 + |f(x_n)|^2) + \theta(1 - \theta)\Delta^2(|f(y_n)|^2 + |f(x_n)|^2). \end{array} \tag{2.9}$$

Then, by $\Delta < 1, 0 \leq \theta \leq 1$ and (2.2), we derive that

$$\begin{array}{l} |y_n|^2 \leq (1 + \Delta)|x_n|^2 + (1 + \Delta)\theta\Delta(\alpha + \beta|y_n|^2) + (1 - \theta)(1 + \Delta)\Delta(\alpha + \beta|x_n|^2) \\ \quad \leq (2 + 2\beta\Delta)|x_n|^2 + 2\beta\Delta|y_n|^2 + 4\alpha\Delta. \end{array} \tag{2.10}$$

Taking expectation on both sides of (2.10), we receive

$$\mathbb{E}[|y_n|^2] \leq (2+2\beta\Delta)\mathbb{E}[|x_n|^2] + 2\beta\Delta\mathbb{E}[|y_n|^2] + 4\alpha\Delta.$$

Note that the assumption of $2\beta\Delta \leq 1/2$, we take maximum on the left hand of the above inequality and then obtain (2.6). □

Theorem 2.1. *Let x_n and y_n be produced by (1.2). Assume that f, $g : \mathbb{R}^d \mapsto \mathbb{R}^d$ satisfy (2.2) and (2.3). If $\Delta < \min\{1, 1/(4\beta)\}$, then*

$$\max_{0\leq i\leq n} \mathbb{E}[|x_i|^p] \leq C, \qquad \max_{0\leq i\leq n} \mathbb{E}[|y_i|^p] \leq C, \tag{2.11}$$

where C are positive constants.

Proof. It follows from (1.2) that

$$x_{n+1} = y_n + g(y_n)\Delta w_n + \frac{1}{2}L^1 g(y_n)(|\Delta w_n|^2 - \Delta). \tag{2.12}$$

Squaring both sides of (2.12), we obtain

$$\begin{aligned} |x_{n+1}|^2 = \;& |y_n|^2 + |g(y_n)\Delta w_n|^2 + \tfrac{1}{4}|L^1 g(y_n)(|\Delta w_n|^2 - \Delta)|^2 \\ & + \langle g(y_n)\Delta w_n, L^1 g(y_n)(|\Delta w_n|^2 - \Delta)\rangle \\ & + 2\langle y_n, g(y_n)\Delta w_n + \tfrac{1}{2}L^1 g(y_n)(|\Delta w_n|^2 - \Delta)\rangle. \end{aligned} \tag{2.13}$$

Using the inequality $x^T y \leq |x|^2 + |y|^2$, then

$$\begin{aligned} |x_{n+1}|^2 \leq \;& |y_n|^2 + 2|g(y_n)\Delta w_n|^2 + \tfrac{1}{2}|L^1 g(y_n)(|\Delta w_n|^2 - \Delta)|^2 \\ & + 2\langle y_n, g(y_n)\Delta w_n\rangle + \langle y_n, L^1 g(y_n)(|\Delta w_n|^2 - \Delta)\rangle, \end{aligned} \tag{2.14}$$

which implies

$$\begin{aligned} |x_{n+1}|^2 \leq \;& |y_0|^2 + 2\sum_{i=0}^{n} |g(y_i)|^2|\Delta w_i|^2 + \tfrac{1}{2}\sum_{i=0}^{n} |L^1 g(y_i)|^2(|\Delta w_i|^2 - \Delta)^2 \\ & + 2\sum_{i=0}^{n}\langle y_i, g(y_i)\Delta w_i\rangle + \sum_{i=0}^{n}\langle y_i, L^1 g(y_i)(|\Delta w_i|^2 - \Delta)\rangle. \end{aligned} \tag{2.15}$$

Recall the elementary inequality: if $x_1, ..., x_n \geq 0$, $p \geq 1$, and n is a positive integer, then

$$\left(\sum_{i=1}^{n} x_i\right)^p \leq n^{p-1}\sum_{i=1}^{n} x_i^p. \tag{2.16}$$

Form Equation (2.16), we therefore have

$$\begin{aligned} \frac{|x_{n+1}|^p}{5^{p-1}} \leq \;& |y_0|^{2p} + 2^p\left(\sum_{i=0}^{n}|g(y_i)|^2|\Delta w_i|^2\right)^p + \left|\sum_{i=0}^{n}\langle y_i, L^1 g(y_i)(|\Delta w_i|^2 - \Delta)\rangle\right|^p \\ & + 2^p\left|\sum_{i=0}^{n}\langle y_i, g(y_i)\Delta w_i\rangle\right|^p + 2^{-p}\left(\sum_{i=0}^{n}|L^1 g(y_i)|^2(|\Delta w_i|^2 - \Delta)^2\right)^p. \end{aligned} \tag{2.17}$$

Now, we will estimate terms Equation (2.17) separately. From [14], we have

$$\mathbb{E}\left(\sup_{0\leq k\leq n}\sum_{i=0}^{k}|g(y_i)|^2|\Delta w_i|^2\right)^p \leq C + C\Delta\sum_{i=0}^{n}\mathbb{E}[|y_i|^{2p}], \tag{2.18}$$

$$\mathbb{E}\left[\sup_{0\leq k\leq n}\left|\sum_{i=0}^{k} y_i^T L^1 g(y_i)(|\Delta w_i|^2 - \Delta)\right|^p\right] \leq C + C\Delta\sum_{i=0}^{n}\mathbb{E}[|y_i|^{2p}], \tag{2.19}$$

$$\mathbb{E}\left[\sup_{0\le k\le n}\left|\sum_{i=0}^{k} y_i^T g(y_i)\Delta w_i\right|^p\right] \le C + C\Delta\sum_{i=0}^{n}\mathbb{E}[|y_i|^{2p}] \tag{2.20}$$

and

$$\mathbb{E}\left(\sup_{0\le k\le n}\sum_{i=0}^{k}|L^1 g(y_i)|^2(|\Delta w_i|^2-\Delta)^2\right)^p \le C + C\Delta\sum_{i=0}^{n}\mathbb{E}[|y_i|^{2p}]. \tag{2.21}$$

Combining (2.18)-(2.21) with (2.17) yields

$$\mathbb{E}\left[\sup_{0\le k\le n+1}|x_k|^{2p}\right] \le C + C\Delta\sum_{i=0}^{n}\mathbb{E}\left[\sup_{0\le k\le i}|y_k|^{2p}\right]. \tag{2.22}$$

From Lemma 2.1, we deduce that

$$\mathbb{E}\left[\sup_{0\le k\le n+1}|x_n|^{2p}\right] \le C + C\Delta\sum_{i=0}^{n}\mathbb{E}\left[\sup_{0\le k\le i}|x_k|^{2p}\right]. \tag{2.23}$$

Using the discrete-type Gronwall inequality and noting that $(n+1)\Delta \le T$ give

$$\mathbb{E}\left[\sup_{0\le k\le n+1}|x_n|^{2p}\right] \le C. \tag{2.24}$$

This together with (2.6) gives the desired assertion (2.11). □

3 Convergence of the SSTM-method

Lemma 3.1. *Suppose that $f:\mathbb{R}^d \mapsto \mathbb{R}^d$ satisfies (2.2) and (2.3). If $\Delta < \min\{1, 1/(4\beta)\}$, then the x_n and y_n produced by (1.2) satisfy*

$$\mathbb{E}[|x_n - y_n|^2] \le C\Delta^2, \qquad \mathbb{E}[|x_{n+1} - y_n|^2] \le C\Delta, \tag{3.1}$$

where $0 \le n \le N$ and C is a positive constant.

Proof. From (1.2) we have

$$y_n - x_n = \theta f(y_n)\Delta + (1-\theta)f(x_n)\Delta.$$

Hence, it follows from the inequality $2ab \le a^2 + b^2$ and $0 \le \theta \le 1$ that

$$\begin{aligned} |y_n - x_n|^2 &\le 2\Delta^2(|f(y_n)|^2 + |f(x_n)|^2) \\ &\le 2\Delta^2(\alpha + \beta|y_n|^2 + \alpha + \beta|x_n|^2). \end{aligned} \tag{3.2}$$

Taking expectation on both sides of the above inequality, and using the estimate (2.11) in Theorem 2.1. We have

$$\mathbb{E}[|x_n - y_n|^2] \le C\Delta^2. \tag{3.3}$$

It follows from (1.2) that

$$x_{n+1} - y_n = g(y_n)\Delta w_n + \frac{1}{2}L^1 g(y_n)(|\Delta w_n|^2 - \Delta).$$

Squaring on both sides, we have

$$\begin{aligned} |x_{n+1} - y_n|^2 &\le 2|g(y_n)\Delta w_n|^2 + \tfrac{1}{4}\left|L^1 g(y_n)(|\Delta w_n|^2 - \Delta)\right|^2 \\ &\le 2(\alpha + \beta|y_n|^2)|\Delta w_n|^2 + \tfrac{1}{4}(\alpha + \beta|y_n|^2)(|\Delta w_n|^2 - \Delta)^2. \end{aligned}$$

Taking expectation and using (2.11), we have

$$\mathbb{E}[|x_{n+1} - y_n|^2] \le C\Delta. \tag{3.4}$$

□

The proof of the following lemma is similar to that of Theorem 4.1 in [15].

Let $\delta_{n+1} := x(t_{n+1}) - \widetilde{x}_{n+1}$, $\varepsilon_{n+1} := x(t_{n+1}) - x_{n+1}$, where

$$\begin{aligned}\widetilde{y}_n &= x(t_n) + \Delta((1-\theta)f(x(t_n)) + \theta f(\widetilde{y}_n)),\\ \widetilde{x}_{n+1} &= \widetilde{y}_n + g(\widetilde{y}_n)\Delta W_n + \tfrac{1}{2}L^1 g(\widetilde{y}_n)(|\Delta w_n|^2 - \Delta).\end{aligned}$$

Lemma 3.2. *Let Assumption 1 hold and the local error of the SSTM-method (1.2) satisfies*

$$\max_{0\le n\le N-1} |\mathbb{E}[\delta_{n+1}]| \le C\Delta^{p_1}, \qquad as \quad \Delta \to 0$$

and

$$\max_{0\le n\le N-1} (\mathbb{E}[\delta_{n+1}^2])^{1/2} \le C\Delta^{p_2}, \qquad as \quad \Delta \to 0$$

where $p_2 \ge 1/2$ *and* $p_1 \ge p_2 + 1/2$. *Then the estimate*

$$\max_{1\le n\le N} (\mathbb{E}[\varepsilon_n^2])^{1/2} \le C\Delta^{p_2 - 1/2}$$

holds, where the constant C is independent of Δ *but dependent on the length of the time interval T and initial segment.*

According to the definitions of $\widetilde{y}_n$ and $\widetilde{x}_{n+1}$, we have $\mathbb{E}[y_n|x_n = x(t_n)] = \mathbb{E}[\widetilde{y}_n]$ and $\mathbb{E}[x_{n+1}|x_n = x(t_n)] = \mathbb{E}[\widetilde{x}_{n+1}]$.

Now we can prove the mean-square convergence of the STM-method.

Theorem 3.1. *Suppose that* f, $g : \mathbb{R}^d \mapsto \mathbb{R}^d$ *satisfy conditions (2.2) and (2.3). Let* $x(t)$ *be the exact solution of Eq.(1.1) and* x_n *be the approximate solution produced by the SSTM-method (1.2). Then there exists a positive constant C such that*

$$\max_{1\le n\le N} \mathbb{E}[|x(t_n) - x_n|^2] \le C\Delta, \qquad \Delta \to 0.$$

Proof. Define the θ-Milstein Scheme ([10]) as follows

$$x_{n+1}^M = x_n + \theta f(x_{n+1})\Delta + (1-\theta)f(x_n)\Delta + g(x_n)\Delta w_n + \frac{1}{2}L^1 g(x_n)(|\Delta w_n|^2 - \Delta).$$

Due to Theorem 4.1 in ([10]), then there exists constant C, which may vary at each line, such that

$$\begin{aligned}\mathbb{E}|\delta_{n+1}|^2 &= \mathbb{E}\left(|x(t_{n+1}) - x_{n+1}|^2 | x_n = x(t_n)\right)\\ &\le 2\mathbb{E}\left(|x(t_{n+1}) - x_{n+1}^M|^2 | x_n = x(t_n)\right) + 2A_1\\ &\le C\Delta^2 + 2A_1,\end{aligned}$$

$$\begin{aligned}A_1 &= \mathbb{E}\left(|x_{n+1}^M - x_{n+1}|^2 | x_n = x(t_n)\right)\\ &= \mathbb{E}(|\Delta\theta(f(x_{n+1}) - f(y_n)) + (g(x_n) - g(y_n))\Delta w_n\\ &\quad + \tfrac{1}{2}(L^1 g(x_n) - L^1 g(y_n))((\Delta w_n)^2 - \Delta)|^2 | x_n = x(t_n))\\ &\le 3\Delta^2\mathbb{E}\left(|f(x_{n+1}) - f(y_n)|^2 | x_n = x(t_n)\right)\\ &\quad + 3\Delta\mathbb{E}\left(|g(x_n) - g(y_n)|^2 | x_n = x(t_n)\right)\\ &\quad + 3\Delta\mathbb{E}\left(|L^1 g(x_n) - L^1 g(y_n)|^2 | x_n = x(t_n)\right)\\ &\le 3K\Delta^2\mathbb{E}\left(|x_{n+1} - y_n|^2 | x_n = x(t_n)\right)\\ &\quad + 3\mu\Delta\mathbb{E}\left(|x_n - y_n|^2 | x_n = x(t_n)\right)\\ &\quad + 3\delta\Delta\mathbb{E}\left(|x_n - y_n|^2 | x_n = x(t_n)\right)\\ &\le C\Delta^3.\end{aligned} \tag{3.5}$$

Inequality (3.5) is easy to be checked by the Assumption 1 and Lemma 2. Then

$$\begin{aligned} |\mathbb{E}[\delta_{n+1}]| &= |\mathbb{E}[x(t_{n+1}) - x_{n+1}|x_n = x(t_n)]| \\ &\leq |\mathbb{E}[x(t_{n+1}) - x^M_{n+1}|x_n = x(t_n)]| + A_2 \\ &\leq C\Delta^{3/2} + A_2, \end{aligned} \tag{3.6}$$

where

$$\begin{aligned} A_2 &= |\mathbb{E}[x^M_{n+1} - x_{n+1}|x_n = x(t_n)]| \\ &\leq (\mathbb{E}[(x^M_{n+1} - x_{n+1})^2|x_n = x(t_n)])^2 \\ &\leq C\Delta^{3/2}, \end{aligned} \tag{3.7}$$

which follows from the inequality $|\mathbb{E}[x]| \leq \mathbb{E}|x| \leq (\mathbb{E}[x^2])^{1/2}$. Due to Lemma 3.1, we obtain $\max_{1\leq n\leq N} \mathbb{E}[(x_n - x(t_n))^2|x_0 = x(t_0)] = O(\Delta))$, the Theorem 3.1 is obtained. □

4 Mean Square Stability of the SSTM-method

In this section, we focus on the nonlinear stability of schemes (1.2). To this end, we first assume that $f(0) = g(0) = 0$, and then from (2.1)-(2.3), we obtain

$$|f(x)|^2 \leq K|x|^2, \quad |g(x)|^2 \leq \mu|x|^2, \quad |L^1g(x)|^2 \leq \delta|x|^2. \tag{4.1}$$

Theorem 4.1. *Assume that the condition (4.1) hold. If there exists a positive constants $\lambda > \mu$ such that for all $x \in \mathbb{R}^n$,*

$$2x^T f(x) \leq -\lambda|x|^2. \tag{4.2}$$

Then for any $\Delta \leq \Delta^* = \frac{\lambda-\mu}{(1-\theta)K+1/2\delta}$, the SSTM-method is mean-square stable.

Proof. It follows from (1.2) that

$$y_n - \theta f(y_n)\Delta = x_n + \Delta(1-\theta)f(x_n).$$

Squaring both sides of the above equality, we have

$$|y_n|^2 - 2\theta\Delta y_n^T f(y_n) + \Delta^2\theta^2|f(y_n)|^2 = |x_n|^2 + 2\Delta(1-\theta)x_n^T f(x_n) + \Delta^2(1-\theta)^2|f(x_n)|^2$$

and

$$|y_n|^2 \leq |x_n|^2 + 2\theta\Delta y_n^T f(y_n) + 2\Delta(1-\theta)x_n^T f(x_n) + \Delta^2(1-\theta)^2|f(x_n)|^2. \tag{4.3}$$

Taking expectation on both side of (4.3), it follows from conditions (4.1) and (4.2) that

$$\mathbb{E}[|y_n|^2] \leq \mathbb{E}[|x_n|^2] - \theta\Delta\lambda\mathbb{E}[|y_n|^2] - \Delta(1-\theta)\lambda\mathbb{E}[|x_n|^2] + \Delta^2(1-\theta)^2K\mathbb{E}[|x_n|^2]. \tag{4.4}$$

It follows from (1.2) that

$$x_{n+1} = y_n + g(y_n)\Delta w_n + \frac{1}{2}L^1g(y_n)(|\Delta w_n|^2 - \Delta). \tag{4.5}$$

Squaring both sides of (4.5) and takeing expectation, we obtain

$$\begin{aligned} \mathbb{E}[|x_{n+1}|^2] &= \mathbb{E}[|y_n|^2] + \mathbb{E}[|g(y_n)\Delta W_n|^2] + \tfrac{1}{4}\mathbb{E}[|L^1g(y_n)(|\Delta w_n|^2 - \Delta)|^2] \\ &\quad +2\mathbb{E}[\langle y_n + \tfrac{1}{2}L^1g(y_n)(|\Delta w_n|^2 - \Delta), g(y_n)\Delta w_n\rangle] \\ &\quad +\mathbb{E}[y_n^T L^1g(y_n)(|\Delta w_n|^2 - \Delta)]. \end{aligned} \tag{4.6}$$

Noting that x_n is $\mathcal{F}_{t_n}$-measurable at mesh point t_n. So

$$\begin{aligned} &\mathbb{E}[|g(y_n)\Delta w_n|^2] = \Delta\mathbb{E}[|g(y_n)|^2], \\ &\mathbb{E}[|L^1g(y_n)(|\Delta w_n|^2 - \Delta)|^2] = 2\Delta^2\mathbb{E}[|L^1g(y_n)|^2], \\ &\mathbb{E}[\langle y_n + \tfrac{1}{2}L^1g(y_n)(|\Delta w_n|^2 - \Delta), g(y^n)\Delta w_n\rangle] = 0, \\ &\mathbb{E}[y_n^T L^1g(y_n)(|\Delta w_n|^2 - \Delta)] = 0. \end{aligned}$$

Then it follows from the (4.6) that

$$\mathbb{E}[|x_{n+1}|^2] = \mathbb{E}[|y_n|^2] + \Delta\mathbb{E}[|g(y_n)|^2] + \frac{1}{2}\Delta^2\mathbb{E}[|L^1 g(y_n)|^2].$$

We can induce from the above equality that

$$\mathbb{E}[|y_n|^2] \leq \mathbb{E}[|x_{n+1}|^2]. \tag{4.7}$$

Form the condition (4.1), we deduce that

$$\mathbb{E}[|x_{n+1}|^2] \leq \mathbb{E}[|y_n|^2] + \mu\Delta\mathbb{E}[|y_n|^2] + \frac{1}{2}\delta\Delta^2\mathbb{E}[|y_n|^2]. \tag{4.8}$$

Applying the inequality (4.8) to the first term of the right side of (4.4), we obtain

$$\begin{aligned}\mathbb{E}[|y_n|^2] \quad & \leq \mathbb{E}[|y_{n-1}|^2] + \theta\Delta(-\lambda\mathbb{E}[|y_n|^2]) + \Delta(1-\theta)(-\lambda\mathbb{E}[|x_n|^2]) \\ & \quad +\Delta^2(1-\theta)^2 K\mathbb{E}[|x_n|^2] + \mu\Delta\mathbb{E}[|y_{n-1}|^2] + \tfrac{1}{2}\delta\Delta^2\mathbb{E}[|y_{n-1}|^2] \\ & = \mathbb{E}[|y_{n-1}|^2] + \theta\Delta(-\lambda\mathbb{E}[|y_n|^2]) + \Delta(1-\theta)(-\lambda\mathbb{E}[|x_n|^2]) \\ & \quad +\Delta^2(1-\theta)^2 K\mathbb{E}[|x_n|^2] + \theta\Delta\mu\mathbb{E}[|y_{n-1}|^2] + (1-\theta)\Delta\mu\mathbb{E}[|y_{n-1}|^2] \\ & \quad +\tfrac{1}{2}\theta\Delta^2\delta\mathbb{E}[|y_{n-1}|^2] + \tfrac{1}{2}(1-\theta)\Delta^2\delta\mathbb{E}[|y_{n-1}|^2].\end{aligned}$$

By the estimation (4.7), it is obtained that

$$\begin{aligned}\mathbb{E}[|y_n|^2] \quad & \leq \mathbb{E}[|y_{n-1}|^2] + \theta\Delta(-\lambda\mathbb{E}[|y_n|^2]) + \Delta(1-\theta)(-\lambda\mathbb{E}[|x_n|^2] \\ & \quad +\Delta^2(1-\theta)^2 K\mathbb{E}[|x_n|^2] + \theta\Delta\mu\mathbb{E}[|y_{n-1}|^2] + (1-\theta)\Delta\mu\mathbb{E}[|x_n|^2] \\ & \quad +\tfrac{1}{2}\theta\Delta^2\delta\mathbb{E}[|y_{n-1}|^2] + \tfrac{1}{2}(1-\theta)\Delta^2\delta\mathbb{E}[|x_n|^2].\end{aligned}$$

Namely

$$\begin{aligned}\mathbb{E}[|y_n|^2] - \mathbb{E}[|y_{n-1}|^2] \quad & \leq \theta\Delta(-\lambda\mathbb{E}[|y_n|^2] + (\mu + \tfrac{1}{2}\Delta\delta)\mathbb{E}[|y_{n-1}|^2]) \\ & \quad +\Delta(1-\theta)(-\lambda + \Delta(1-\theta)K + \mu + \tfrac{1}{2}\Delta\delta)\mathbb{E}[|x_n|^2].\end{aligned} \tag{4.9}$$

Summation of (4.9) over n from $n = 1$ gives

$$\begin{aligned}& \mathbb{E}[|y_n|^2] + \Delta(1-\theta)(\lambda - \Delta(1-\theta)K - \mu - \tfrac{1}{2}\Delta\delta)\sum_{i=1}^{n}\mathbb{E}[|x_i|^2] \\ & +\theta\Delta(\lambda - \mu - \tfrac{1}{2}\Delta\delta)\sum_{i=1}^{n}\mathbb{E}[|y_i|^2]) \leq (1 + (\mu + 1/2\Delta\delta)\theta\Delta)\mathbb{E}[|y_0|^2].\end{aligned} \tag{4.10}$$

Let $\Delta^* = \frac{\lambda-\mu}{(1-\theta)K+1/2\delta}$. For $\Delta \leq \Delta^*$, the condition $\lambda - \Delta(1-\theta)K - \mu - \frac{1}{2}\Delta\delta \geq 0$ hold. Then the series $\sum\mathbb{E}(y_i)^2$ and $\sum\mathbb{E}(x_i)^2$ are bound to be convergent. So we obtain that

$$\lim_{n\to\infty}\mathbb{E}[x_n^2] = 0. \tag{4.11}$$

□

Remark 4.1. *The conditions in Theorem 4.1 are not optimal because the SSTM method may still be stable for some stepsizes Δ which are larger than Δ^*.*

5 Numerical Examples

In this section, we compare computational efficiency and stability properties of the split-step theta Milstein scheme, (θ, σ)- Milstein scheme [10] and the tamed Milstein scheme [9]. In order to estimate the rate of convergence we proceed with numerical experiments for the following SDEs. We focus on root mean-square errors

$$\left(\mathbb{E}|x(T) - x_N|^2\right)^{1/2} < \varepsilon, \tag{5.1}$$

where $\varepsilon > 0$, $N = T/\Delta$ and the expectation is approximated by computing sample average over 2000 paths. We choose a simple scalar equation

$$\begin{cases} \mathrm{d}x(t) = -x(t)\mathrm{d}t + x(t)\mathrm{d}w(t), & t \in [0, T] \\ x(0) = 1. \end{cases} \tag{5.2}$$

By some estimation we arrive at $K = \mu = \delta = 1$, $\lambda = 2$. From Fig.1, we observe that SSTM scheme is consistent with strong order of convergence equal to one. As expected, the SSTM scheme gives an error that less than the other two models. Fig.2 illustrates the mean square stability of numerical solution obtained by the SSTM method when $\Delta = 1.5$, $\theta = 0$. But we can see that the tamed Milstein scheme is unstable for equal stepsize. In this case, the restriction on stepsize Δ of mean square stable SSTM method is less than that of the tamed Milstein scheme.

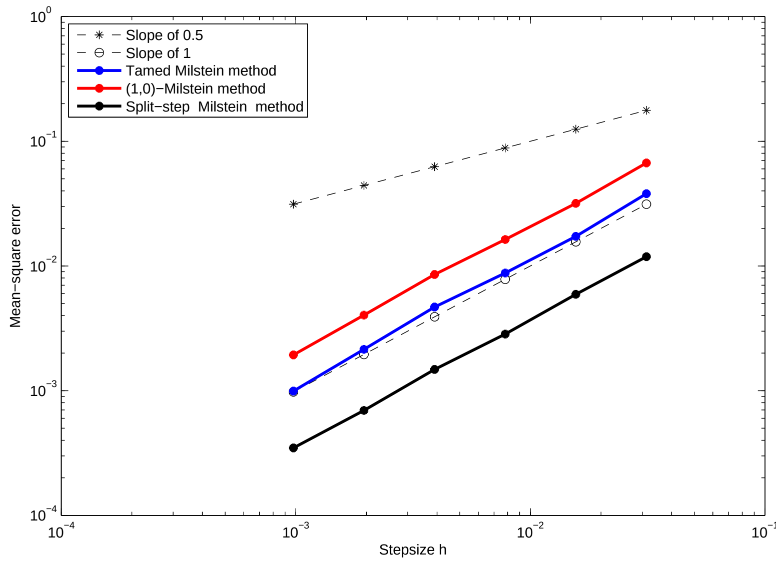

Fig. 1. Root mean square approximation error versus stepsize Δ to approximate (5.2)

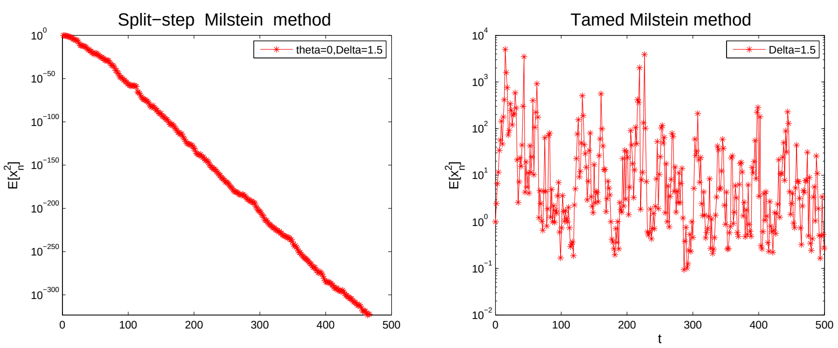

Fig. 2. Comparison of mean square stabilities between different Milstein methods to approximate (5.2)

6 Conclusion

In this work, we have examined the mean square stability of the SSTM-scheme for SDEs under the local Lipschitz condition for the drift and diffusion coefficients. We have proved that SSTM-method is convergent with order 1 in mean-square sense. Numerical tests verify the relationship that theory predicts between the parameter θ and the step size h for mean-square stability of the SDEs.

Competing Interests

Authors have declared that no competing interests exist.

References

[1] Bodo BA, Thompson ME, Unny TE. A review on stochastic differential equations for applications in hydrology. Stochastic Hydrology and Hydraulics. 1987;1(2):81-100.

[2] Zuhra S, et al. Solving singular boundary value problems by optimal homotopy asymptotic method. International Journal of Differential Equations. 2014;2014.

[3] Ullah H, et al. The three dimensional flow past a stretching sheet by extended optimal homotopy asymptotic method. Science International. Science International. 2014;26:567-576.

[4] Van Kampen, Nicolaas Godfried. Stochastic processes in physics and chemistry. Elsevier. 1992;1.

[5] Wilkinson Darren J. Stochastic modelling for systems biology. CRC press; 2011.

[6] Gillespie Daniel T. Stochastic simulation of chemical kinetics. Annu. Rev. Phys. Chem. 2007;58:35-55.

[7] Platen Eckhard, Nicola Bruti-Liberati. Numerical solution of stochastic differential equations with jumps in finance. Springer Science & Business Media. 2010;64.

[8] Mil'šhteĭn GN. Approximate integration of stochastic differential equations. Theory of Probability & Its Applications. 1975;19(3):557-562.

[9] Wang, Xiaojie, Siqing Gan. The tamed Milstein method for commutative stochastic differential equations with non-globally Lipschitz continuous coefficients. Journal of Difference Equations and Applications. 2013;19(3):466-490.

[10] Higham Desmond J, Xuerong Mao, Lukasz Szpruch. Convergence, non-negativity and stability of a new Milstein scheme with applications to finance. Discrete Contin. Dyn. Syst. Ser. B. 2013;18(8):2083-2100.

[11] Wang, Zhiyong, Chengjian Zhang. An analysis of stability of Milstein method for stochastic differential equations with delay. Computers & Mathematics with Applications. 2006;51(9):1445-1452.

[12] Tocino A, Senosiain MJ. Two-step Milstein schemes for stochastic differential equations. Numerical Algorithms. 2015;69(3):643-665.

[13] Khasminskii R. Stochastic Stability of Differential Equations[J]. Stochastic Modelling & Applied Probability. 1980;66.

[14] Jiang Fengze, et al. Double-implicit and split two-step Milstein schemes for stochastic differential equations. International Journal of Computer Mathematics. 2015;1-25.

[15] Baker Christopher TH, Evelyn Buckwar. Numerical analysis of explicit one-step methods for stochastic delay differential equations. LMS Journal of Computation and Mathematics. 2000;3:315-335.

Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)
http://sciencedomain.org/review-history/17638