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Research Article

Controllability of Mild Solution of Nonlocal Conformable Fractional Differential Equations

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In many research works Bouaouid et al. have proved the existence of mild solutions of an abstract class of nonlocal conformable fractional Cauchy problem of the form: $d^{\alpha}x(t)/dt^{\alpha} = Ax(t) + f(t, x(t)), x(0) = x_0 + g(x), t \in [0, \tau]$. The present paper is a continuation of these works in order to study the controllability of mild solution of the above Cauchy problem. Precisely, we shall be concerned with the controllability of mild solution of the following Cauchy problem $d^{\alpha}x(t)/dt^{\alpha} = Ax(t) + f(t, x(t)) + Bu(t), x(0) = x_0 + g(x), t \in [0, \tau]$, where $d^{\alpha}(.)/dt^{\alpha}$ is the vectorial conformable fractional derivative of order $\alpha \in]0, 1]$ in a Banach space X and A is the infinitesimal generator of a semigroup $(T(t))_{t\geq 0}$ on X. The element x_0 is a fixed vector in X and f, g are given functions. The control function u is an element of $L^2([0, \tau], U)$ with U is a Banach space and B is a bounded linear operator from U into X.

1. Introduction

Mathematical models based on factional derivatives with respect to time have been the focus of many studies due to their recent applications in various areas of science [1–5]. Many concrete applications prove that the fractional derivative is a very good approaches to deal better with modeling of dynamical systems with memories [6–17]. Regarding to the literature of fractional calculus, it is well known that there are many approaches to define fractional derivatives including the Riemann-Liouville and Caputo definitions. Unfortunately, these definitions have some shortcomings. For example, they do not satisfy derivative formulas for the product and quotient of two functions. In consequence, many researchers have paid attention to propose a best and simple definition of fractional derivative [18, 19]. For example in the work [18], the authors have proposed a new definition of fractional derivative named conformable fractional derivative. This novel fractional derivative is very simple and verifies all the properties of the classical derivative. Actually, the conformable fractional derivative becomes the subject of many research contributions [20–39].

For example in [20–22], the authors have proved the existence of mild solution for the following nonlocal conformable fractional Cauchy problem:

$$\frac{d^{\alpha}x(t)}{dt^{\alpha}} = Ax(t) + f(t, x(t)), x(0) = x_0 + g(x), t \in [0, \tau], \quad (1)$$

where $d^{\alpha}(.)/dt^{\alpha}$ represents the conformable fractional derivative of order $\alpha \in]0, 1]$, and A is the infinitesimal generator of a semigroup $(T(t))_{t\geq 0}$ on a Banach space $(X, \|.\|)$ ([40]). The element x_0 is a fixed vector in X and $f : [0, \tau] \times X \longrightarrow X$, $g : \mathscr{C} \longrightarrow X$ are given functions, with $\mathscr{C}$ is the Banach space of continuous functions $x(.)$ defined from $[0, \tau]$ into X equipped with the norm $|x|_c = \sup_{t\in[0,\tau]}\|x(t)\|$. The expression $x(0) = x_0 + g(x)$ means the so-called nonlocal condition, which can be applied in physics with better effects than the classical initial condition [41–43].

Recently, the study of control problems has attracted the attention of many mathematicians and physicists in various fields of science [44–50]. For example, in theory of differential equations, the controllability consists to control evolution systems from the initial position to the desired position. Motivated by the fact that the controllability is a most important qualitative behavior of a dynamical system, we will be concerned with the controllability of the Cauchy problem (1). Precisely, we will prove a controllability result for the following Cauchy problem

$$\frac{d^\alpha x(t)}{dt^\alpha} = Ax(t) + f(t, x(t)) + Bu(t), x(0) = x_0 + g(x), t \in [0, \tau], \tag{2}$$

where the control function $u(.)$ is an element of $L^2([0, \tau], U)$ with U is a Banach space and B is a bounded linear operator from U into X.

The rest of this paper is organized as follows. In Section 2, we briefly recall some tools related to the conformable fractional calculus. In Section 3, we present the main result. Section 4 is devoted to a concert application.

2. Preliminaries

Recalling some preliminary facts on the conformable fractional calculus.

Definition 1 (see [18]). For $\alpha \in]0, 1]$, the conformable fractional derivative of order α of a function x(.): $[0,+\infty[\longrightarrow \mathbb{R}$ is defined as

$$\begin{aligned} \frac{d^\alpha x(t)}{dt^\alpha} &= \lim_{\varepsilon \longrightarrow 0} \frac{x\left(t + \varepsilon t^{1-\alpha}\right) - x(t)}{\varepsilon} \text{ for } t > 0 \text{ and } \frac{d^\alpha x(0)}{dt^\alpha} \\ &= \lim_{t \longrightarrow 0^+} \frac{d^\alpha x(t)}{dt^\alpha}, \end{aligned} \tag{3}$$

provided that the limits exist.

The conformable fractional integral $I^\alpha(.)$ of a function $x(.)$ is defined by

$$I^\alpha(x)(t) = \int_0^t s^{\alpha-1} x(s) ds, \text{ for } t > 0. \tag{4}$$

Theorem 2 (see [21]). *If $x(.)$ is a continuous function in the domain of $I^\alpha(.)$, then, we have*

$$\frac{d^\alpha (I^\alpha(x)(t))}{dt^\alpha} = x(t). \tag{5}$$

Theorem 3 (see [23]). *If $x(.)$ is a differentiable function, then, we have*

$$I^\alpha \left(\frac{d^\alpha x(.)}{dt^\alpha} \right)(t) = x(t) - x(0). \tag{6}$$

Definition 4 (see [23]). The conformable fractional Laplace transform of order $\alpha \in]0, 1]$ of a function $x(.)$ is defined as follows

$$\mathscr{L}_\alpha(x(t))(\lambda) \coloneqq \int_0^{+\infty} t^{\alpha-1} e^{-\lambda t^\alpha / \alpha} x(t) dt, \lambda > 0. \tag{7}$$

The following proposition gives us the actions of the conformable fractional integral and the conformable fractional Laplace transform on the conformable fractional derivative, respectively.

Proposition 5 (see [23]). *If $x(.)$ is a differentiable function, then, we have the following results*

$$I^\alpha \left(\frac{d^\alpha x(.)}{dt^\alpha} \right)(t) = x(t) - x(0), \tag{8}$$

$$\mathscr{L}_\alpha \left(\frac{d^\alpha x(t)}{dt^\alpha} \right)(\lambda) = \lambda \mathscr{L}_\alpha(x(t))(\lambda) - x(0). \tag{9}$$

According to [28], we have the following remark.

Remark 6. For two functions $x(.)$ and $y(.)$, we have

$$\mathscr{L}_\alpha \left(x\left(\frac{t^\alpha}{\alpha} \right) \right)(\lambda) = \mathscr{L}_1(x(t))(\lambda), \tag{10}$$

$$\mathscr{L}_\alpha \left(\int_0^t s^{\alpha-1} x\left(\frac{t^\alpha - s^\alpha}{\alpha} \right) y(s) ds \right)(\lambda) = \mathscr{L}_1(x(t))(\lambda) \mathscr{L}_\alpha(y(t))(\lambda), \tag{11}$$

provided that the both terms of each equality exist.

3. Main Result

Lemma 7. *If $x \in \mathscr{C}$ is a solution of Cauchy problem (2), then, the function $x(.)$ satisfies the following integral equation*

$$\begin{aligned} x(t) &= T\left(\frac{t^\alpha}{\alpha} \right)[x_0 + g(x)] \\ &+ \int_0^t s^{\alpha-1} T\left(\frac{t^\alpha - s^\alpha}{\alpha} \right)(f(s, x(s)) + Bu(s)) ds. \end{aligned} \tag{12}$$

The proof of this result is essentially based on the conformable fractional Laplace transform. For the complete proof, one can see the works [20–22].

Definition 8 (see [20–22]). A function $x \in \mathscr{C}$ is called a mild solution of Cauchy problem (2) if

$$\begin{aligned} x(t) &= T\left(\frac{t^\alpha}{\alpha} \right)[x_0 + g(x)] \\ &+ \int_0^t s^{\alpha-1} T\left(\frac{t^\alpha - s^\alpha}{\alpha} \right)(f(s, x(s)) + Bu(s)) ds. \end{aligned} \tag{13}$$

Now, we deal with the controllability of Cauchy problem (2).

Definition 9. The Cauchy problem (2) is said to be controllable on $[0,\tau]$, if for every $x_1 \in X$, there exists a control $u \in L^2([0,\tau],U)$ such that the mild solution $x(.)$ of (2) satisfies $x(\tau)=x_1$.

In the sequel of this paper, we will need the following assumptions:

(H_1) The function $f(t,.)$: $X \longrightarrow X$ is continuous and there exist positive constants L, K such that $\|f(t,x)\| \le L\|x\|$ and $\|f(t,y)-f(t,x)\| \le K\|y-x\|$ for all $x,y\in X$.

(H_2) The function $f(.,x)$: $[0,\tau] \longrightarrow X$ is continuous for all $x \in X$.

(H_3) The function $g:\mathscr{C} \longrightarrow X$ is continuous.

(H_4) There exist positive constants M and N such that

$$\|g(x)\| \le M|x|_c \text{ and } \|g(y)-g(x)\| \le N|y-x|_c \text{for all } x,y\in\mathscr{C}. \quad (14)$$

(H_5) The bounded linear operator $W: L^2([0,\tau],U) \longrightarrow X$ defined by

$$W(u)=\int_0^\tau s^{\alpha-1}T\left(\frac{\tau^\alpha - s^\alpha}{\alpha}\right)Bu(s)ds, \quad (15)$$

has an induced inverse operator $\tilde{W}^{-1}$, which takes values in $L^2([0,\tau],U)/\text{Ker (W)}$, and there exist positive constants R_1, R_2 such that $\|B\| \le R_1$ and $\|\tilde{W}^{-1}\| \le R_2$.

Theorem 10. *Assume that $(H_1)-(H_5)$ hold, then Cauchy problem (2) is controllable on $[0,\tau]$, provided that*

$$\begin{aligned}&\sup_{t\in[0,\tau]}\left|T\left(\frac{t^\alpha}{\alpha}\right)\right|\left(1+R_1R_2\frac{\tau^\alpha}{\alpha}\sup_{t\in[0,\tau]}\left|T\left(\frac{\tau^\alpha}{\alpha}\right)\right|\right)\\ &\cdot\max\left(M+\frac{\tau^\alpha}{\alpha}L, N+\frac{\tau^\alpha}{\alpha}K\right)<1.\end{aligned} \quad (16)$$

Proof. By using hypothesis (H_5) for an arbitrary function $x(.)$, we can define a control $u_x(.)$ as follows

$$\begin{aligned}u_x(.)=\tilde{W}^{-1}&\left(x_1-T\left(\frac{\tau^\alpha}{\alpha}\right)[x_0+g(x)]\right.\\ &\left.-\int_0^\tau s^{\alpha-1}T\left(\frac{\tau^\alpha-s^\alpha}{\alpha}\right)f(s,x(s))ds\right)(.).\end{aligned} \quad (17)$$

For this control, we define the operator $\Psi:\mathscr{C}\longrightarrow\mathscr{C}$ by

$$\begin{aligned}\Psi(x)(t)=&T\left(\frac{t^\alpha}{\alpha}\right)[x_0+g(x)]\\ &+\int_0^t s^{\alpha-1}T\left(\frac{t^\alpha-s^\alpha}{\alpha}\right)(f(s,x(s))+Bu_x(s))ds.\end{aligned} \quad (18)$$

□

We also introduce for a radius $r>0$ the ball $B_r := \{x\in\mathscr{C}, |x|_c \le r\}$, and we denote by $|.|$ the norm in the space $\mathscr{L}(X)$ of bounded operators defined from X into itself.

We will show that the operator Ψ has a fixed point, which is a mild solution of the control problem (2). To do so, we will give the proof in two steps.

Step 1. Prove that there exists a radius $\delta>0$ such that $\Gamma: B_\delta \longrightarrow B_\delta$.

For $x\in\mathscr{C}$ and $t\in[0,\tau]$, we have

$$\begin{aligned}\Psi(x)(t)=&T\left(\frac{t^\alpha}{\alpha}\right)[x_0+g(x)]\\ &+\int_0^t s^{\alpha-1}T\left(\frac{t^\alpha-s^\alpha}{\alpha}\right)(f(s,x(s))+Bu_x(s))ds.\end{aligned} \quad (19)$$

Then, one has

$$\begin{aligned}\|\Psi(x)(t)\| \le &\sup_{t\in[0,\tau]}\left|T\left(\frac{t^\alpha}{\alpha}\right)\right|\left[\|x_0+g(x)\|\right.\\ &\left.+\int_0^t s^{\alpha-1}\|f(s,x(s))+Bu_x(s)\|ds\right].\end{aligned} \quad (20)$$

By using hypothesis (H_1), (H_4), and (H_5), we obtain

$$\begin{aligned}\|\Psi(x)(t)\| \le &\sup_{t\in[0,\tau]}\left|T\left(\frac{t^\alpha}{\alpha}\right)\right|\left[\|x_0\|+M|x|_c\right.\\ &\left.+\left(L|x|_c+R_1\|u_x\|_2\right)\int_0^\tau s^{\alpha-1}ds\right]\\ \le &\sup_{t\in[0,\tau]}\left|T\left(\frac{t^\alpha}{\alpha}\right)\right|\left[\|x_0\|+M|x|_c\right.\\ &\left.+\left(L|x|_c+R_1\|u_x\|_2\right)\frac{\tau^\alpha}{\alpha}\right].(*).\end{aligned} \quad (21)$$

On the other hand, we have known that

$$\begin{aligned}u_x=\tilde{W}^{-1}&\left(x_1-T\left(\frac{\tau^\alpha}{\alpha}\right)[x_0+g(x)]\right.\\ &\left.-\int_0^\tau s^{\alpha-1}T\left(\frac{\tau^\alpha-s^\alpha}{\alpha}\right)f(s,x(s))ds\right).\end{aligned} \quad (22)$$

In view of assumptions (H_1), (H_4), and (H_5), we obtain

$$\begin{aligned}\|u_x\|_2 \le R_2&\left\|x_1-T\left(\frac{\tau^\alpha}{\alpha}\right)[x_0+g(x)]\right.\\ &\left.-\int_0^\tau s^{\alpha-1}T\left(\frac{\tau^\alpha-s^\alpha}{\alpha}\right)f(s,x(s))ds\right\|\\ \le R_2&\left[\|x_1\|+\sup_{t\in[0,\tau]}\left|T\left(\frac{t^\alpha}{\alpha}\right)\right|\right.\\ &\left.\cdot\left(\|x_0+g(x)\|+\int_0^\tau s^{\alpha-1}\|f(s,x(s))\|ds\right)\right]\end{aligned}$$

$$\le R_2\left[\|x_1\| + \sup_{t\in[0,\tau]}\left|T\left(\frac{t^\alpha}{\alpha}\right)\right| \cdot \left(\|x_0\| + M|x|_c + L|x|_c\int_0^\tau s^{\alpha-1}ds\right]\right.$$
$$\le R_2\left[\|x_1\| + \sup_{t\in[0,\tau]}\left|T\left(\frac{t^\alpha}{\alpha}\right)\right|\left(\|x_0\| + M|x|_c + L|x|_c\frac{\tau^\alpha}{\alpha}\right]\right.$$
$$\le R_2\left[\|x_1\| + \sup_{t\in[0,\tau]}\left|T\left(\frac{t^\alpha}{\alpha}\right)\right|\left(\|x_0\| + \left(M + L\frac{\tau^\alpha}{\alpha}\right)|x|_c\right]\right.. \tag{23}$$

By replacing this estimate in $(*)$, we get

$$\|\Psi(x)(t)\| \le \sup_{t\in[0,\tau]}\left|T\left(\frac{t^\alpha}{\alpha}\right)\right|\left[\|x_0\| + M|x|_c + \left(L|x|_c + R_1R_2\left[\|x_1\| + \sup_{t\in[0,\tau]}\left|T\left(\frac{t^\alpha}{\alpha}\right)\right| \cdot \left(\|x_0\| + \left(M + L\frac{\tau^\alpha}{\alpha}\right)|x|_c\right]\right)\frac{\tau^\alpha}{\alpha}\right]. \tag{24}$$

Separating the terms containing the expression $|x|_c$, one has

$$\|\Psi(x)(t)\| \le \sup_{t\in[0,\tau]}\left|T\left(\frac{t^\alpha}{\alpha}\right)\right|\left[M + L\frac{\tau^\alpha}{\alpha} + R_1R_2\frac{\tau^\alpha}{\alpha}\sup_{t\in[0,\tau]}\left|T\left(\frac{t^\alpha}{\alpha}\right)\right|\left(M + L\frac{\tau^\alpha}{\alpha}\right)\right]|x|_c + \sup_{t\in[0,\tau]}\left|T\left(\frac{t^\alpha}{\alpha}\right)\right|\left[\|x_0\| + \frac{\tau^\alpha}{\alpha}R_1R_2\|x_1\| + \frac{\tau^\alpha}{\alpha}R_1R_2\sup_{t\in[0,\tau]}\left|T\left(\frac{t^\alpha}{\alpha}\right)\right|\|x_0\|\right]. \tag{25}$$

By using a simple factorization, we obtain

$$\|\Psi(x)(t)\| \le \sup_{t\in[0,\tau]}\left|T\left(\frac{t^\alpha}{\alpha}\right)\right|\left(M + L\frac{\tau^\alpha}{\alpha}\right) \cdot \left[1 + R_1R_2\frac{\tau^\alpha}{\alpha}\sup_{t\in[0,\tau]}\left|T\left(\frac{t^\alpha}{\alpha}\right)\right|\right]|x|_c + \sup_{t\in[0,\tau]}\left|T\left(\frac{t^\alpha}{\alpha}\right)\right|\left[\left(1 + \frac{\tau^\alpha}{\alpha}R_1R_2\sup_{t\in[0,\tau]}\left|T\left(\frac{t^\alpha}{\alpha}\right)\right|\right) \cdot \|x_0\| + \frac{\tau^\alpha}{\alpha}R_1R_2\|x_1\|\right]. \tag{26}$$

Hence, it suffices to consider δ as a solution in r of the following inequality

$$\sup_{t\in[0,\tau]}\left|T\left(\frac{t^\alpha}{\alpha}\right)\right|\left(M + L\frac{\tau^\alpha}{\alpha}\right)\left[1 + R_1R_2\frac{\tau^\alpha}{\alpha}\sup_{t\in[0,\tau]}\left|T\left(\frac{t^\alpha}{\alpha}\right)\right|\right]r + \sup_{t\in[0,\tau]}\left|T\left(\frac{t^\alpha}{\alpha}\right)\right|\left[\left(1 + \frac{\tau^\alpha}{\alpha}R_1R_2\sup_{t\in[0,\tau]}\left|T\left(\frac{t^\alpha}{\alpha}\right)\right|\right)\|x_0\| + \frac{\tau^\alpha}{\alpha}R_1R_2\|x_1\|\right] \le r. \tag{27}$$

Precisely, we can choose δ such that

$$\delta \ge \frac{\sup_{t\in[0,\tau]}|T(t^\alpha/\alpha)|\left[\left(1 + (\tau^\alpha/\alpha)R_1R_2\sup_{t\in[0,\tau]}|T(t^\alpha/\alpha)|\right)\|x_0\| + (\tau^\alpha/\alpha)R_1R_2\|x_1\|\right]}{1 - \sup_{t\in[0,\tau]}|T(t^\alpha/\alpha)|(M + L(\tau^\alpha/\alpha))\left[1 + R_1R_2(\tau^\alpha/\alpha)\sup_{t\in[0,\tau]}|T(t^\alpha/\alpha)|\right]}. \tag{28}$$

Step 2. We show that Ψ is a contraction operator on B_δ. For $y, x\in\mathscr{C}$, we have

$$\Psi(y)(t) - \Psi(x)(t) = T\left(\frac{t^\alpha}{\alpha}\right)[g(y) - g(x)] + \int_0^t s^{\alpha-1}T\left(\frac{t^\alpha - s^\alpha}{\alpha}\right)(f(s, y(s)) - f(s, x(s)) + B(u_y - u_x)(s))ds. \tag{29}$$

According to (H_1), (H_4), and (H_5), we obtain

$$\|\Psi(y)(t) - \Psi(x)(t)\| \le \sup_{t\in[0,\tau]}\left|T\left(\frac{t^\alpha}{\alpha}\right)\right|\left[\|g(y) - g(x)\| + \int_0^t s^{\alpha-1}\left\|f(s, y(s)) - f(s, x(s)) + B(u_y - u_x)(s)\right\|ds\right]$$
$$\le \sup_{t\in[0,\tau]}\left|T\left(\frac{t^\alpha}{\alpha}\right)\right|\left[N|y - x|_c + \left(K|y - x|_c + R_1\left\|u_y - u_x\right\|_2\right)\int_0^t s^{\alpha-1}ds\right]$$
$$\le \sup_{t\in[0,\tau]}\left|T\left(\frac{t^\alpha}{\alpha}\right)\right|\left[N|y - x|_c + \frac{\tau^\alpha}{\alpha}\left(K|y - x|_c + R_1\left\|u_y - u_x\right\|_2\right)\right].(**). \tag{30}$$

In the other hand, we know that

$$u_y - u_x = \tilde{W}^{-1}\left(-T\left(\frac{\tau^\alpha}{\alpha}\right)[g(y)-g(x)] - \int_0^\tau s^{\alpha-1}T\left(\frac{\tau^\alpha - s^\alpha}{\alpha}\right) \cdot (f(s,y(s)) - f(s,x(s)))ds\right). \tag{31}$$

Then, one has

$$\begin{aligned}\|u_y - u_x\|_2 &\le R_2 \sup_{t\in[0,\tau]}\left|T\left(\frac{t^\alpha}{\alpha}\right)\right|\left[\|g(y)-g(x)\| + \int_0^\tau s^{\alpha-1}\|f(s,y(s)) - f(s,x(s)))\|ds\right.\\ &\le R_2 \sup_{t\in[0,\tau]}\left|T\left(\frac{t^\alpha}{\alpha}\right)\right|\left[N + K\frac{\tau^\alpha}{\alpha}\right]|y-x|_c.\end{aligned} \tag{32}$$

By replacing this estimate in $(**)$, we obtain

$$\begin{aligned}&\|\Psi(y)(t) - \Psi(x)(t)\|\\ &\le \sup_{t\in[0,\tau]}\left|T\left(\frac{t^\alpha}{\alpha}\right)\right|\left[N|y-x|_c + \frac{\tau^\alpha}{\alpha}\left(K|y-x|_c + R_1R_2\sup_{t\in[0,\tau]}\left|T\left(\frac{t^\alpha}{\alpha}\right)\right|\left(N + K\frac{\tau^\alpha}{\alpha}\right)|y-x|_c\right)\right]\\ &\le \sup_{t\in[0,\tau]}\left|T\left(\frac{t^\alpha}{\alpha}\right)\right|\left[N + \frac{\tau^\alpha}{\alpha}K + \frac{\tau^\alpha}{\alpha}R_1R_2\sup_{t\in[0,\tau]}\left|T\left(\frac{t^\alpha}{\alpha}\right)\right|\left(N + K\frac{\tau^\alpha}{\alpha}\right)\right]|y-x|_c\\ &\le \sup_{t\in[0,\tau]}\left|T\left(\frac{t^\alpha}{\alpha}\right)\right|\left(N + \frac{\tau^\alpha}{\alpha}K\right) \cdot \left[1 + \frac{\tau^\alpha}{\alpha}R_1R_2\sup_{t\in[0,\tau]}\left|T\left(\frac{t^\alpha}{\alpha}\right)\right|\right]|y-x|_c.\end{aligned} \tag{33}$$

Taking the supremum, we get

$$|\Psi(y)(t) - \Psi(x)|_c \le \sup_{t\in[0,\tau]}\left|T\left(\frac{t^\alpha}{\alpha}\right)\right|\left(N + \frac{\tau^\alpha}{\alpha}K\right) \cdot \left[1 + \frac{\tau^\alpha}{\alpha}R_1R_2\sup_{t\in[0,\tau]}\left|T\left(\frac{t^\alpha}{\alpha}\right)\right|\right]|y-x|_c. \tag{34}$$

Since $\sup_{t\in[0,\tau]}|T(t^\alpha/\alpha)|(N + t^\alpha/\alpha K)[1 + t^\alpha/\alpha R_1R_2 \sup_{t\in[0,\tau]}|T(t^\alpha/\alpha)|] < 1$, then, Ψ is a contraction operator on B_δ. Hence, there exists a unique element $x_\delta(.) \in B_\delta$ such that $\Psi(x_\delta)(t) = x_\delta(t)$ for all $t \in [0,\tau]$.

It remains to show that the mild solution x_δ is controllable. To this end, we have

$$\begin{aligned}x_\delta(\tau) = \Psi(x_\delta)(\tau) &:= T\left(\frac{\tau^\alpha}{\alpha}\right)[x_0 + g(x_\delta)] + \int_0^\tau s^{\alpha-1}T\left(\frac{\tau^\alpha - s^\alpha}{\alpha}\right)\left(f(s,x_\delta(s)) + Bu_{x_\delta}(s)\right)ds\\ &= T\left(\frac{\tau^\alpha}{\alpha}\right)[x_0 + g(x_\delta)] + \int_0^\tau s^{\alpha-1}T\left(\frac{\tau^\alpha - s^\alpha}{\alpha}\right)f(s,x_\delta(s))ds + \int_0^\tau s^{\alpha-1}T\left(\frac{\tau^\alpha - s^\alpha}{\alpha}\right)Bu_{x_\delta}(s)ds\\ &= -W(x_\delta) + x_1 + \int_0^\tau s^{\alpha-1}T\left(\frac{\tau^\alpha - s^\alpha}{\alpha}\right)Bu_{x_\delta}(s)ds\\ &= -W(x_\delta) + x_1 + W(x_\delta)\\ &= x_1.\end{aligned} \tag{35}$$

Thus, Cauchy problem (2) is controllable on $[0,\tau]$.

4. Application

Let $X = U = L^2([0,1])$ be equipped with the inner product and norm defined by $<u,v> = \sqrt{\int_0^1 u(s)\bar{v}(s)ds}$ and $\|u\| = \sqrt{\int_0^1 |u(s)|^2 ds}$. Define the operator A by

$$A\varphi = \ddot{\varphi}, \tag{36}$$

$$D(A) = \{\varphi \in X, \varphi, \dot{\varphi} \text{ are absolutely continous and } \ddot{\varphi} \in X, \varphi(0) = \varphi(1) = 0\}. \tag{37}$$

As well known, the operator A has a discrete spectrum, and the eigenvalues are $\{-n^2, n \in \mathbb{N}\}$ with the corresponding normalized eigenvectors $x_n(y) = \sqrt{2}\sin(ny)$, $n = 1, 2, \cdots$. The operator A generates a contraction semigroup $(T(t))_{t\ge 0}$ given explicitly by

$$T(t)x = \sum_{n=1}^{+\infty} e^{-n^2 t} < x, x_n > x_n, x \in X. \tag{38}$$

Then, we have

$$Ax = -\sum_{n=1}^{+\infty} n^2 < x, x_n > x_n, x \in D(A). \tag{39}$$

Next, define the control operator B as follows

$$B(u) = \sum_{n=1}^{+\infty} e^{-1/n^2+1} < u, x_n > x_n, u \in U. \tag{40}$$

We have

$$\|B(u)\| = \sum_{n=1}^{+\infty} e^{-2/n^2+1} < u, x_n >^2 \leq \sum_{n=1}^{+\infty} < u, x_n >^2 \leq \|u\|. \quad (41)$$

Then, $\|B\| \leq 1$ and thus the operator B is bounded. Now return back to the operator W, we obtain

$$\begin{aligned} W(u) &= \int_0^1 s^{\alpha-1} T\left(\frac{\tau^\alpha - s^\alpha}{\alpha}\right) Bu(s) ds \\ &= \sum_{n=1}^{+\infty} \frac{\left(1 - e^{-n^2/\alpha}\right) e^{-1/n^2+1}}{n^2} < u, x_n > x_n. \end{aligned} \quad (42)$$

Hence, the right inverse of the operator W may defined as follows

$$W^{-1} : D(A) \longrightarrow L^2\left([0, 1], L^2([0, 1])\right), \quad (43)$$

$$u \mapsto W^{-1}(u) = \sum_{n=1}^{+\infty} \frac{n^2 e^{1/n^2+1}}{1 - e^{-n^2/\alpha}} < u, x_n > x_n. \quad (44)$$

For the operator W^{-1}, we get

$$\begin{aligned} \left\| W^{-1}(u) \right\| &\leq \frac{e^{1/2}}{1 - e^{-1/\alpha}} \sqrt{\sum_{n=1}^{+\infty} n^4 < u, x_n > x_n} = \frac{e^{1/2}}{1 - e^{-1/\alpha}} \|Au\| \\ &= \frac{e^{1/2}}{1 - e^{-1/\alpha}} \|u\|_{D(A)}. \end{aligned} \quad (45)$$

Define the functions $f : [0, 1] \times X \longrightarrow X$ and $g : X \longrightarrow X$ by

$$f(t, x(t)) = \frac{e^{-t} |x(t)|}{(20 + e^t)(1 + |x(t)|)}, \quad (46)$$

$$g(x) = \sum_{i=1}^{n} a_i x(t_i) \text{ where } \sum_{i=1}^{n} |a_i| \leq \frac{1}{20} \text{ and } 0 < t_1 < t_2 < \cdots < t_n < 1. \quad (47)$$

For the function f, we have

$$\begin{aligned} \|f(t, x) - f(t, y)\| &= \frac{e^{-t}}{20 + e^t} \left\| \frac{x}{1 + x} - \frac{y}{1 + y} \right\| \\ &\leq \frac{e^{-t}}{20 + e^t} \|x - y\| \\ &\leq \frac{1}{20} \|x - y\|. \end{aligned} \quad (48)$$

Here in this application example, we have $\tau = 1$, $L = 1/20$, $K = 1/20$, $M = 1/20$, $N = 1/20$, $R_1 = 1$, $R_2 = e^{1/2}/1 - e^{-1/\alpha}$ and $\sup_{t\in[0,1]} |T(t^\alpha/\alpha)| \leq 1$. Then, the contraction condition assumed in Theorem 10 becomes

$$\begin{aligned} &\sup_{t\in[0,\tau]} \left| T\left(\frac{t^\alpha}{\alpha}\right) \right| \left(1 + R_1 R_2 \frac{\tau^\alpha}{\alpha} \sup_{t\in[0,\tau]} \left| T\left(\frac{\tau^\alpha}{\alpha}\right) \right| \right) \\ &\cdot \max\left(M + \frac{\tau^\alpha}{\alpha} L, N + \frac{\tau^\alpha}{\alpha} K\right) \leq \frac{1 + \alpha}{20\alpha^2} \left(\alpha + \frac{e^{1/2}}{1 - e^{-1/\alpha}}\right). \end{aligned} \quad (49)$$

For $\alpha = 1/2$ in the last contraction condition, we get $1 + \alpha/20\alpha^2(\alpha + e^{1/2}/1 - e^{-1/\alpha}) \simeq 0.72 < 1$. Thus, by using Theorem 10, we conclude that the following Cauchy problem

$$\begin{cases} \dfrac{\partial^{1/2} x(t)}{\partial t^{1/2}} = Ax(t) + \dfrac{e^{-t} |x(t)|}{(20 + e^t)(1 + |x(t)|)} + Bu, t \in [0, 1], \\ x(0) = \displaystyle\sum_{i=1}^{n} a_i x(t_i). \end{cases} \quad (50)$$

Has a unique controllable mild solution.

5. Conclusion and Comments

The existence of mild solutions of a Cauchy problem of nonlocal differential equations with conformable fractional derivative is largely studied in several works [20–22]. Our contribution in this present work is the study of the controllability of mild solutions for such Cauchy problems by means of the Banach fixed point theorem combined with theory of semigroups of linear operators. We notice that the constants of increases of the norms of the bounded operators W and W^{-1} in the previous application are given directly in a simple way in terms of the exponential function, however, for the Caputo fractional derivative in the application of the nice work [51] are given in terms of the so-called Mittag-Leffler function.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares no conflicts of interest.

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