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Research Article

The K Extended Laguerre Polynomials Involving $\left\{ A_{r,n,k}^{(\alpha)}(x) \right\}$ ${}_rF_r, r>2$

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In this manuscript, we present the generalized hypergeometric function of the type ${}_rF_r, r>2$ and extension of the K Laguerre polynomial for the K extended Laguerre polynomials $\{A_{r,n,k}^{(\alpha)}(x)\}$. Additionally, we describe the K generating function, K recurrence relations, and KS Rodrigues formula.

1. Introduction

Laguerre polynomials are utilized to investigate non-central Chi-square distribution. Many works are existed in the literature with implementation to classical orthogonal polynomials. There many extensions of Laguerre polynomials.

A large number of properties of Laguerre polynomials have been described in classical works, e.g., Erdélyi et al. [1] and Bell [2]; also we can refer to Wang and Guo [3] and Mathai [4].

Chak [5] has given a representation for the Laguerre polynomials. Carlitz [6] proved the recurrence relations involving Laguerre polynomials. Al-Salam [7] proved several results involving Laguerre polynomials. Prabhakar [8] introduced that generating functions, integrals, and recurrence relations are obtained for the polynomials $Z_n^\alpha(x;k)$ in x^k..

Andrews et al. [9], Chen and Srivastava [10], Trickovic and Stankovic [11], Radulescu [12], and Doha et al. [13] have done a lot of work for properties of Laguerre polynomials. Akbary et al. [14] can be referred for other application of Laguerre polynomials. Li [15], Aksoy et al. [16], Wang [17], and Krasikov and Zarkh [18] studied problems of permutation of polynomials; bijection that can induce polynomials with integer coefficients is modulo m.

In this manuscript, we present the properties of the extending Laguerre polynomial including ${}_rF_r, r>2$; we consider

$$L_n^{(\alpha)}(x) = \frac{(1+\alpha)_n}{n!}\,{}_1F_1(-n;1+\alpha;x). \tag{1}$$

Shively [19] extended the Laguerre polynomials as

$$R_n(a,x) = \frac{(a)_{2n}}{n!(a)_n}\,{}_1F_1(-n;a+n;x). \tag{2}$$

Habibullah [20] demonstrated the Rodrigues formula as

$$\begin{aligned} R_n(a+1,x) &= \frac{e^x x^{-\alpha-n}}{n!} D^n\left(x^{\alpha+2n}e^{-x}\right), \\ L_n^{(\alpha)}(x) &= \frac{e^x x^{-\alpha}}{n!} D^n\left(x^{\alpha+n}e^{-x}\right). \end{aligned} \tag{3}$$

Erdélyi et al. [1] introduced

$$D^m\left[x^{\alpha+m}L_n^{(\alpha+m)}(x)\right] = \frac{\Gamma(\alpha+m+n+1)}{\Gamma(\alpha+n+1)}x^\alpha L_n^{(\alpha)}(x), D = \frac{d}{dx}. \tag{4}$$

Khan and Habibullah [21] introduced $A_{2,n}(x) = {}_2F_2(-n/2, (-n+1/2) ; 1/2, 1 ; x^2)$.

Khan and Kalim [22] introduced

$$A_{3,m}^{(\alpha)}(y) = \frac{(1+\alpha)_m}{m!}\,{}_3F_3\left(\frac{-m}{3}, \frac{-m+1}{3}, \frac{-m+2}{3} ; \frac{1+\alpha}{3}, \frac{2+\alpha}{3}, \frac{3+\alpha}{3} ; y^3\right). \quad (5)$$

Khan et al. [23] proposed extended Laguerre polynomials $\left\{A \begin{matrix} (\alpha) \\ q, n \end{matrix} (x)\right\}$.

Parashar [24] presented a new set of Laguerre polynomials $L_n^{(\alpha,h)}(x)$ related to the Laguerre polynomials $L_n^{(\alpha)}(x)$. Sharma and Chongdar [25] proved an extension of bilateral generating functions of the modified Laguerre polynomials.

Researchers [26–28] found additional properties of k gamma and k beta functions. Then, Mubeen and Habibullah [29] introduced k fractional integrals and discussed its application. Mubeen and Habibullah [30] introduced an integral representation of some k hypergeometric functions. Krasniqi [31] derived some properties of the k gamma and k beta function. Mubeen [32] proved the properties of confluent k integrals by using k fractional integrals. There is a tremendous scope to study k polynomials using k gamma, k beta, and k hypergeometric functions. Kokologiannaki and Krasniqi [33] introduced k analogue of the Riemann Zeta function and also proved some inequities relating to Riemann Zeta function and k gamma functions.

Din et al. [34] understand the dynamical behavior such diseases; they fitted a susceptible-infectious quarantined model for human cases with constant proportions. Din et al. [35] investigated a newly constructed system of equation for hepatitis B disease in sense of Atanganaa–Baleanu Caputo (ABC) fractional order derivative. Din et al. [36] developed the analysis of a non-integer-order model for hepatitis B (HBV) under singular type Caputo fractional order derivative. They investigated proposed system for an approximate or semi-analytical solution using Laplace transform along with decomposition techniques by Adomian polynomial of nonlinear terms and some perturbation techniques of homotopy (HPM). Din [37] investigated the spread of such contagion by using a delayed stochastic epidemic model with general incidence rate, time-delay transmission, and the concept of cross immunity.

Ain et al. [38] impression of activated charcoal is shaped by the fractional dynamics of the problem, which leads to speedy and low-cost first aid. Ain et al. [39] presented an impulsive differential equation system, which is useful in examining the effectiveness of activated charcoal in detoxifying the body with methanol poisoning. Din and Ain [40] developed a model based on a stochastic process that could be utilized to portray the effect of arbitrary-order derivatives. A nonlinear perturbation is used to study the proposed stochastic model with the help of white noises.

Rehman et al.'s [41] unsaturated porous media were analyzed by solving Burger's equation using the variational iterative modeling and homotopy perturbation method. Wang and Wang [42] described two different types of plasma models with variable coefficients by using the fractal derivative. Wang [43] investigated the fractal nonlinear dispersive Boussineq-like equation by variational perspective for the first time. The fractal variational principle of the fractal Boussineq-like equation was established via fractal semi-inverse method (FSM).

2. Extended Polynomials

Lemma 1.

If $k, j \in \mathbb{Z}^+$ and n is any non-negative integer. Then, we will get

$$\left(\frac{-n}{r}\right)_{kj}\left(\frac{-n+1}{r}\right)_{kj}\cdots\left(\frac{-n+r-1}{r}\right)_{kj} = (-1)^{rkj}\frac{n!}{r^{rkj}(n-rkj)!}. \quad (6)$$

Proof.

$$\begin{aligned}
&\left(\frac{-n}{r}\right)_{kj}\left(\frac{-n+1}{r}\right)_{kj}\cdots\left(\frac{-n+r-1}{r}\right)_{kj} \\
&= \left(\frac{-n}{r}\right)\left(\frac{-n}{r}+1\right)\left(\frac{-n}{r}+2\right)\cdots \\
&\left(\frac{-n}{r}+kj-1\right)\left(\frac{-n+1}{r}\right)\left(\frac{-n+1}{r}+1\right)\left(\frac{-n+1}{r}+2\right)\cdots \\
&\left(\frac{-n+1}{r}+kj-1\right)\left(\frac{-n+r-1}{r}\right)\left(\frac{-n+r-1}{r}+1\right) \\
&\left(\frac{-n+r-1}{r}+2\right)\cdots\left(\frac{-n+r-1}{r}+kj-1\right) \\
&= \left(\frac{-n}{r}\right)\left(\frac{-n+r}{r}\right)\left(\frac{-n+2r}{r}\right)\cdots\left(\frac{-n+rkj-r}{r}\right)\left(\frac{-n+1}{r}\right) \\
&\left(\frac{-n+r+1}{r}\right)\left(\frac{-n+2r+1}{r}\right)\cdots\left(\frac{-n+rkj-r+1}{r}\right) \\
&\left(\frac{-n+r-1}{r}\right)\left(\frac{-n+2r-1}{r}\right)\left(\frac{-n+3r-1}{r}\right)\cdots\left(\frac{-n+rkj-1}{r}\right).
\end{aligned} \quad (7)$$

By simplification we get our desired result. □

Lemma 2.

If $k \in \mathbb{Z}^+$ and n is any non-negative integer, thus

$$(\alpha)_{kn} = k^{kn}\left(\frac{\alpha}{k}\right)_n\left(\frac{\alpha+1}{k}\right)_n\cdots\left(\frac{\alpha+k-1}{k}\right)_n. \quad (8)$$

Rainville [44] (p 22)).

Lemma 3.

Assume that $k \in \mathbb{Z}^+$ and n is any non-negative integer. Then, we reach

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} B(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} B(k, n+k). \quad (9)$$

Rainville [44] (p 57)).

Table 1: .

The extended Laguerre polynomials $A_{q,n}^{(\alpha)}(x)$ Khan et al. [23]
The K extended Laguerre polynomials $A_{r,n,k}^{(\alpha)}(x)$
$A_{q,n}^{(\alpha)}(x)=\frac{e^x(q+\alpha)_n}{n!}\,{}_qF_q\left(\begin{matrix}\frac{-n}{q},\frac{-n+1}{q},\cdots,\frac{-n+q-1}{q};\\ \frac{q+\alpha}{q},\frac{q+1+\alpha}{q},\cdots,\frac{2q+\alpha-1}{q}\end{matrix};x^q\right)$
$A_{r,n,k}^{(\alpha)}(x)=\frac{e^x(rk+\alpha)_{n,k}}{(n;k)!}\,{}_rF_{r,k}\left(\begin{matrix}\left(\frac{-n}{r},k\right),\left(\frac{-n+k}{r},k\right),\cdots,\left(\frac{-n+rk-1}{r},k\right);\\ \left(\frac{\alpha+kr}{r},k\right),\left(\frac{\alpha+rk+1}{r},k\right),\cdots,\left(\frac{\alpha+2rk-1}{r},k\right)\end{matrix};x^r\right).$
If we put $k=1$ in our paper, then we get the result of Khan et al. [23].

Lemma 4.

Assume that $k\in\mathbb{Z}^+$ and n is any non-negative integer. Thus, we have

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}B(k,n)=\sum_{n=0}^{\infty}\sum_{k=0}^{n}B(k,n-k). \tag{10}$$

Rainville [44] (p 56)).

3. The K Extended Laguerre Polynomials $A_{r,n,k}^{(\alpha)}(x)$

We describe the K extended Laguerre polynomial set $\{A_{r,n,k}^{(\alpha)}(x)\}$ as

$$A_{r,n,k}^{(\alpha)}(x)=\frac{e^x(rk+\alpha)_{n,k}}{(n;k)!}\,{}_rF_{r,k}\left(\begin{matrix}\left(\frac{-n}{r},k\right),\left(\frac{-n+k}{r},k\right),\cdots,\left(\frac{-n+rk+1}{r},k\right);\\ \left(\frac{\alpha+kr}{r},k\right),\left(\frac{\alpha+rk+1}{r},k\right),\cdots,\left(\frac{\alpha+2rk-1}{r},k\right)\end{matrix};x^r\right), \tag{11}$$

where $\alpha\in\mathbb{R},n,r,k\in\mathbb{Z}^+$.

Theorem 5.

If $\{A_{r,n,k}^{(\alpha)}(x)\}$, are the K extended Laguerre polynomials. Then

$$A_{r,n,k}^{(\alpha)}(x)=e^x(rk+\alpha)_{n,k}\sum_{j=0}^{[n/rk]}\frac{(-1)^{rkj}}{(n-rkj;k)!(rk+\alpha)_{rkj}}\frac{(x)^{rkj}}{(rkj;k)!}, \tag{12}$$

$$\alpha\in\mathbb{R},n,r,k\in\mathbb{Z}^+. \tag{13}$$

Proof.

Consider

$$\begin{aligned}A_{r,n,k}^{(\alpha)}(x)&=\frac{e^x(rk+\alpha)_{n,k}}{(n;k)!}\,{}_qF_{q,k}\left(\begin{matrix}\left(\frac{-n}{r},k\right),\left(\frac{-n+k}{r},k\right),\cdots,\left(\frac{-n+rk+1}{r},k\right);\\ ;x^r\\ \left(\frac{\alpha+rk}{r},k\right),\left(\frac{\alpha+rk+1}{r},k\right),\cdots,\left(\frac{\alpha+2rk-1}{r},k\right)\end{matrix}\right).\\ &=\frac{e^x(rk+\alpha)_{n,k}}{(n;k)!}\times\sum_{j=0}^{[n/rk]}\left\{\frac{((-n/r),k)_j((-n+k/r),k)_j\cdots((-n+rk+1/r),k)_j}{((\alpha+kq/r),k)_j((\alpha+qk+1/r),k)_j\cdots((\alpha+2rk-1/r),k)_j}\right\}\frac{(x)^{rkj}}{(rkj;k)!}.\end{aligned} \tag{14}$$

By using Lemma (1)

$$\begin{aligned}A_{r,n,k}^{(\alpha)}(x)&=\frac{e^x(rk+\alpha)_{n,k}}{(n;k)!}\\ &\quad\times\sum_{j=0}^{[n/rk]}\left[\frac{(-1)^{rkj}(n;k)!}{r^{rj}(n-rkj;k)!((\alpha+rk/r),k)_j((\alpha+rk+1/r),k)_j\cdots((\alpha+2rk-1/r),k)_j}\right]\frac{(x)^{rkj}}{(rkj;k)!}.\end{aligned} \tag{15}$$

Now, by applying Lemma (2), we get our desired result. □

4. K Generating Functions

Theorem 6.

Suppose that $n, j, k \in \mathbb{Z}^+$. Thus, we reach

$$\sum_{n=0}^{\infty}\sum_{j=0}^{[n/rk]}\frac{(-1)^{rkj}e^x t^n}{(n-rkj;k)!(rk+\alpha)_{rkj}}\frac{(x)^{rkj}}{(rkj;k)!} = e^x M_k(t)\,{}_0F_{r,k}\left(--;\left(\frac{rk+\alpha}{r};k\right),\left(\frac{rk+1+\alpha}{r};k\right),\cdots,\left(\frac{2rk+\alpha-1}{r};k\right);\left(\frac{-xt}{r}\right)^r\right). \quad (16)$$

Proof.

We have

$$\begin{aligned}\sum_{n=0}^{\infty}\sum_{j=0}^{[n/rk]}\frac{(-1)^{rkj}e^x t^n}{(n-rkj;k)!(rk+\alpha)_{rkj}}\frac{(x)^{rkj}}{(rkj;k)!} &= \sum_{n=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{rkj}e^x t^{n+rkj}}{(n;k)!(rk+\alpha)_{rkj}}\frac{(x)^{rkj}}{(rkj;k)!}\\ &= e^x\left[\sum_{n=0}^{\infty}\frac{t^n}{(n;k)!}\right]\left[\sum_{j=0}^{\infty}\frac{(-1)^{rkj}t^{rkj}}{(rk+\alpha)_{rkj}}\frac{(x)^{rkj}}{(rkj;k)!}\right]\\ &= e^x M_k(t)\sum_{j=0}^{\infty}\frac{(-xt)^{rkj}}{(rk+\alpha)_{rkj}(rkj;k)!}\end{aligned} \quad (17)$$

By applying Lemma (2), we get

$$\sum_{n=0}^{\infty}\sum_{j=0}^{[n/rk]}\frac{(-1)^{rkj}e^x t^n}{(n-rkj;k)!(rk+\alpha)_{rkj}}\frac{(x)^{rkj}}{(rkj;k)!} = e^x M_k(t) \times \sum_{j=0}^{\infty}\frac{(-xt)^{rkj}}{r^{rkj}((rk+\alpha/r);k)_j((rk+1+\alpha/r);k)_j\cdots((2rk+\alpha-1/r);k)_j(rkj;k)!}. \quad (18)$$

After simplification, we get our result. □

Corollary 7.

Suppose that $\alpha \in \mathbb{R}$and $n, r, j, k \in \mathbb{Z}^+$. Thus, we reach

$$\sum_{n=0}^{\infty}\frac{A^{(\alpha)}_{r,n,k}(x)t^n}{(rk+\alpha)_{n,k}} = e^x M_k(t)\,{}_0F_{r,k}\left(\begin{matrix}--;\\ \left(\frac{-xt}{r}\right)^r\\ \left(\frac{rk+\alpha}{r};k\right),\left(\frac{rk+1+\alpha}{r};k\right),\cdots,\left(\frac{2rk+\alpha-1}{r};k\right);\end{matrix}\right). \quad (19)$$

Proof.

From Equation (12), we acquire

$$\sum_{n=0}^{\infty}\left[\frac{A^{(\alpha)}_{r,n,k}(x)}{(rk+\alpha)_{n,k}}\right]t^n = \sum_{n=0}^{\infty}\left[\sum_{j=0}^{[n/rk]}\left[\frac{(-1)^{rkj}}{(n-rkj;k)!(rk+\alpha)_{rkj}}\right]\frac{(x)^{rkj}}{(rkj;k)!}\right]t^n. \quad (20)$$

Then, we have our result. □

Theorem 8.

If $c \in \mathbb{Z}^+$, then

$$\sum_{n=0}^{\infty}\frac{(c)_{n,k}A^{(\alpha)}_{r,n,k}(x)t^n}{(\alpha+rk)_{n,k}} = \frac{e^x}{(1-kt)_k^{c/k}} \times {}_rF_{r,k}\left(\begin{matrix}\left(\frac{c}{r},k\right),\left(\frac{c+k}{r},k\right),\cdots,\left(\frac{c+rk+1}{r},k\right);\left(\frac{-xt}{(1-kt)_k^{1/k}}\right)^r\\ \left(\frac{\alpha+rk}{r},k\right),\left(\frac{\alpha+rk+1}{r},k\right),\cdots,\left(\frac{\alpha+2rk-1}{r},k\right);\end{matrix}\right). \quad (21)$$

Proof.

From Equation (20), we note that

$$\sum_{n=0}^{\infty}(c)_n\left[\frac{A^{(\alpha)}_{r,n,k}(x)}{(rk+\alpha)_{n,k}}\right]t^n = \sum_{n=0}^{\infty}(c)_n e^x\left[\sum_{j=0}^{[n/rk]}\left[\frac{(-1)^{rkj}}{(n-rkj;k)!(rk+\alpha)_{rkj}}\right]\frac{(x)^{rkj}}{(rkj;k)!}\right]t^n. \quad (22)$$

We get

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(c)_{n,k}A^{(\alpha)}_{r,n,k}(x)t^n}{(rk+\alpha)_{n,k}} &= \sum_{n=0}^{\infty}\sum_{j=0}^{\infty}\frac{(c)_{n+rkj,k}e^x t^{n+rkj}}{(n;k)!}\frac{(-1)^{rj}(x)^{rkj}}{(rk+\alpha)_{rkj,k}(rkj)!}\\ &= \sum_{j=0}^{\infty}\left[\sum_{n=0}^{\infty}\frac{(c+rkj)_{n,k}t^n}{(n;k)!}\right]\left[\frac{(c)_{rj,k}}{(\alpha+rk)_{rj,k}}\right]\frac{e^x(-xt)^{rkj}}{(rkj;k)!},\end{aligned} \quad (23)$$

Since $(c)_{n+rkj,k} = (c+rkj)_{n,k}(c)_{rj,k}$, and $(1-kt)_k^{-(m/k)} = \sum_{n=0}^{\infty}(m)_{n,k}t^n/(n;k)!$ it thus implies that

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(c)_{n,k}A^{(\alpha)}_{r,n,k}(x)t^n}{(\alpha+rk)_{n,k}} &= \sum_{j=0}^{\infty}\left[\frac{(c)_{rkj,k}}{\left[(1-t)^{c+rkj}\right](\alpha+rk)_{rj,k}}\right]\frac{e^x(-xt)^{rkj}}{(rkj;k)!}\\ &= \frac{e^x}{(1-kt)_k^{c/k}}\sum_{j=0}^{\infty}\left[\frac{(c)_{rkj,k}}{(rk+\alpha)_{rj,k}}\right]\frac{\left(-xt/(1-kt)_k^{1/k}\right)^{rjk}}{(rjk;k)!}\end{aligned} \quad (24)$$

□

Corollary 9.

Assume that $\alpha \in \mathbb{R}$ *and* $n, r, j, k \in \mathbb{Z}^+$. *Thus, we reach*

$$\sum_{n=0}^{\infty} A_{r,n,k}^{(\alpha)}(x)t^n = \frac{1}{(1-kt)_k^{(\alpha+qk)/k}} exp\left(\frac{x-2xt}{1-t}\right). \tag{25}$$

Proof.

We choose $c = r + \alpha$ in Equation (21). We can reach the desired results. □

5. K Recurrence Relations

Theorem 10.

Assume that $\alpha \in \mathbb{R}$ *and* $n, j, k \in \mathbb{Z}^+$. *Thus, we reach*

$$xDA_{r,n,k}^{(\alpha)}(x) = (n+x)A_{r,n,k}^{(\alpha)}(x) - (rk+\alpha+n-1)A_{r,n-1,k}^{(\alpha)}(x), D = \frac{d}{dx}. \tag{26}$$

Proof.

From Equation (16)

$$\sum_{n=0}^{\infty} \frac{A_{r,n,k}^{(\alpha)}(x)t^n}{(rk+\alpha)_{n,k}} = e^x M_k(t)\, {}_0F_{r,k}\left(\begin{matrix} --;\left(\frac{-xt}{r}\right)^r \\ \left(\frac{rk+\alpha}{r};k\right), \left(\frac{rk+1+\alpha}{r};k\right), \cdots, \left(\frac{2rk+\alpha-1}{r};k\right); \end{matrix}\right). \tag{27}$$

Let

$$\sigma_{r,n,k}(x) = \frac{A_{r,n,k}^{(\alpha)}(x)}{(\alpha+rk)_{n,k}}. \tag{28}$$

Suppose that

$${}_0F_{r,k}\left(\begin{matrix} --;\left(\frac{-xt}{r}\right)^r \\ \left(\frac{rk+\alpha}{r};k\right), \left(\frac{rk+1+\alpha}{r};k\right), \cdots, \left(\frac{2rk+\alpha-1}{r};k\right); \end{matrix}\right) = \psi\left(\frac{x^r t^r}{r}\right). \tag{29}$$

Then

$$F = e^x M_k(t)\psi\left(\frac{x^r t^r}{r}\right) = \sum_{n=0}^{\infty} \sigma_{r,n,k}(x)t^n. \tag{30}$$

By taking partial derivatives,

$$\frac{\partial F}{\partial x} = e^x M_k(t)\psi + x^{r-1}t^r e^x M_k(t)\psi', \tag{31}$$

$$\frac{\partial F}{\partial t} = e^x M_k(t)\psi + x^r t^{r-1} e^x M_k(t)\psi', \tag{32}$$

$$x\frac{\partial F}{\partial x} - t\frac{\partial F}{\partial t} = xF - tF. \tag{33}$$

Since

$$F = \sum_{n=0}^{\infty} \sigma_{r,n,k}(x)t^n, \tag{34}$$

therefore $\partial F/\partial x = \sum_{n=0}^{\infty} \sigma'_{r,n,k}(x)t^n$, and $t(\partial F/\partial t) = \sum_{n=0}^{\infty} n\, \sigma_{r,n,k}(x)t^n$.

Equation (33), then yields

$$\begin{aligned} &x\sum_{n=0}^{\infty} \sigma'_{r,n,k}(x)t^n - \sum_{n=0}^{\infty} n\sigma_{r,n,k}(x)t^n \\ &\quad = x\sum_{n=0}^{\infty} \sigma_{r,n,k}(x)t^n - \sum_{n=0}^{\infty} \sigma_{r,n,k}(x)t^{n+1} = x\sum_{n=0}^{\infty} \sigma_{r,n,k}(x)t^n \\ &\quad - \sum_{n=1}^{\infty} \sigma_{r,n-1,k}(x)t^n. \end{aligned} \tag{35}$$

We get $\sigma'_{r,0}(x) = 0$, and for $n > 1$, we get our result. □

Theorem 11.

If $\alpha \in \mathbb{R}$ *and* $n \geq 2$, *then*

$$DA_{r,n,k}^{(\alpha)}(x) = DA_{r,n-1,k}^{(\alpha)}(x) + A_{r,n,k}^{(\alpha)}(x) - 2A_{r,n-1,k}^{(\alpha)}(x). \tag{36}$$

Proof.

By (25), we reach

$$(1-t)^{-rk-\alpha}\exp\left[x\left(\frac{1-2t}{1-t}\right)\right] = \sum_{n=0}^{\infty} A_{r,n,k}^{(\alpha)}(x)t^n. \tag{37}$$

Let

$$F = A(t)\exp\left[x\left(\frac{1-2t}{1-t}\right)\right] = \sum_{n=0}^{\infty} y_{r,n,k}(x)t^n, \tag{38}$$

$$\frac{\partial F}{\partial x}=\left(\frac{1-2t}{1-t}\right)A(t)\exp\left[x\left(\frac{1-2t}{1-t}\right)\right], \tag{39}$$

$$(1-t)\frac{\partial F}{\partial x}=(1-2t)A(t)\exp\left[x\left(\frac{1-2t}{1-t}\right)\right]. \tag{40}$$

By using Equation (38), we obtain

$$(1-t)\frac{\partial F}{\partial x}=(1-2t)F. \tag{41}$$

$$\text{Since } F=\sum_{n=0}^{\infty} y_{r,n,k}(x)t^n, \text{ we reach } \frac{\partial F}{\partial x}=\sum_{n=0}^{\infty} y'_{r,n,k}(x)t^n. \tag{42}$$

Equation (41) can be expressed as

$$\sum_{n=0}^{\infty} y'_{r,n,k}(x)t^n-\sum_{n=0}^{\infty} y'_{r,n,k}(x)t^{n+1}=\sum_{n=0}^{\infty} y_{r,n,k}(x)t^n-2\sum_{n=0}^{\infty} y_{r,n,k}(x)t^{n+1}. \tag{43}$$

We get $y'_{r,0,k}(x)=0$, $y'_{r,1,k}(x)=0$, and for $n>2$, we get our result. □

Theorem 12.
If $\alpha\in\mathbb{R}$ and $n\geq r$, then

$$DA_{r,n,k}^{(\alpha)}(x)=A_{r,n,k}^{(\alpha)}(x)-\sum_{j=0}^{n-1} A_{r,j,k}^{(\alpha)}(x). \tag{44}$$

Proof.
We have

$$\frac{\partial F}{\partial x}=\left[1-\frac{t}{1-t}\right]F. \tag{45}$$

Applying Equation (38) yields

$$\frac{\partial F}{\partial x}=\left[1-\frac{t}{1-t}\right]\sum_{n=0}^{\infty} y_{r,n,k}(x)t^n. \tag{46}$$

By using Equation (42), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} y'_{r,n,k}(x)t^n&=\sum_{n=0}^{\infty} y_{r,n,k}(x)t^n-\left[\sum_{n=0}^{\infty} t^{n+1}\right]\left[\sum_{n=0}^{\infty} y_{r,n,k}(x)t^n\right]\\&=\sum_{n=0}^{\infty} y_{r,n,k}(x)t^n-\sum_{n=0}^{\infty}\sum_{j=0}^{\infty} y_{r,j,k}(x)t^j t^{n+1}\end{aligned} \tag{47}$$

By using Lemma (4), we get

$$\begin{aligned}\sum_{n=0}^{\infty} y'_{r,n,k}(x)t^n&=\sum_{n=0}^{\infty} y_{r,n,k}(x)t^n-\sum_{n=0}^{\infty}\sum_{j=0}^{n} y_{r,j,k}(x)t^{n+1}\\&=\sum_{n=0}^{\infty} y_{r,n,k}(x)t^n-\sum_{n=1}^{\infty}\sum_{j=0}^{n-1} y_{r,j,k}(x)t^n\end{aligned} \tag{48}$$

□

Then, we have $y'_{r,0,k}(x)=0$, $y'_{r,1,k}(x)=0$, and for $n>r$,

$$y'_{r,n,k}(x)=y_{r,n,k}(x)-\sum_{j=0}^{n-1} y_{r,j,k}(x). \tag{49}$$

We get our desired result.

Theorem 13.
Suppose that $\alpha\in\mathbb{R}$and $n\geq r+1$. Thus, we get

$$nA_{r,n,k}^{(\alpha)}(x)=(3x-rk-\alpha)A_{r,n-1,k}^{(\alpha)}(x)-(rk+\alpha+n-2)A_{r,n-2,k}^{(\alpha)}(x). \tag{50}$$

Proof.
We have

$$\begin{aligned}&0=nA_{r,n,k}^{(\alpha)}(x)-xDA_{r,n-1,k}^{(\alpha)}\\&(x)+(2x-rk-\alpha-n+1)A_{r,n-1,k}^{(\alpha)}(x), nA_{r,n,k}^{(\alpha)}\\&(x)=xDA_{r,n-1,k}^{(\alpha)}(x)-(2x-rk-\alpha-n+1)A_{r,n-1,k}^{(\alpha)}(x).\end{aligned} \tag{51}$$

Then, after simplification, we get our result. □

Theorem 14.
Assume that $\alpha\in\mathbb{R}$and $n,r,j,k\in\mathbb{Z}^+$. Thus, we obtain

$$A_{r,n-1,k}^{(1+\alpha)}(x)+A_{r,n,k}^{(\alpha)}(x)=A_{r,n,k}^{(1+\alpha)}(x). \tag{52}$$

Proof.
From Equation (12), we obtain

$$\begin{aligned}&A_{r,n-1,k}^{(1+\alpha)}(x)=e^x(rk+1+\alpha)_{n-1,k}\\&\sum_{j=0}^{[(n-1)/rk]}\frac{(-1)^{rkj}}{(n-1-rkj;k)!(r+1+\alpha)_{rkj}}\frac{x^{rkj}}{(rkj;k)!},\end{aligned} \tag{53}$$

so

that-
$A^{(\alpha)}_{r,n,k}(x) = e^x (rk+\alpha)_{n,k} \sum_{j=0}^{[n/rk]} ((-1)^{rkj}/(n-rkj\,;k)!$
$(rk + \alpha)_{rkj})((x)^{rkj}/(rkj\,;k)!)$.

Then, we acquire

$$
\begin{aligned}
&A^{(1+\alpha)}_{r,n-1,k}(x) + A^{(\alpha)}_{r,n,k}(x) = e^x (rk+1+\alpha)_{n-1,k} \sum_{j=0}^{\left[\frac{n-1}{rk}\right]} \frac{(-1)^{rkj}}{(n-1-rkj\,;k)!(r+1+\alpha)_{rkj}} \frac{x^{rkj}}{(rkj\,;k)!} e^x (rk+\alpha)_{n,k} \sum_{j=0}^{\left[\frac{n}{rk}\right]} \frac{(-1)^{rkj}}{(n-rkj\,;k)!(rk+\alpha)_{rkj}} \frac{(x)^{rkj}}{(rkj\,;k)!} \\
&= e^x \left[\sum_{j=0}^{\left[\frac{n-1}{rk}\right]} \frac{(rk+\alpha+n-1)!(-1)^{rkj}}{(n-1-rkj\,;k)!(r+\alpha+rkj)!} \frac{x^{rkj}}{(rkj\,;k)!} + \sum_{k=0}^{\left[\frac{n}{rk}\right]} \frac{(rk+\alpha+n-1)!(-1)^{rkj}}{(n-rkj\,;k)!(r+\alpha+rkj-1)!} \frac{x^{rkj}}{(rkj\,;k)!} \right] \\
&= e^x \left[\sum_{j=0}^{\left[\frac{n-1}{rk}\right]} \frac{(rk+\alpha+n-1)!(-1)^{rkj}}{(n-1-rkj\,;k)!(r+\alpha+rkj)!} \frac{x^{rkj}}{(rkj\,;k)!} + \sum_{k=0}^{\left[\frac{n-1}{rk}\right]} \frac{(rk+\alpha+n-1)!(-1)^{rkj}}{(n-rkj\,;k)!(r+\alpha+rkj-1)!} \frac{x^{rkj}}{(rkj\,;k)!} + \frac{x^{rkn}}{(rkn\,;k)!} \right] \\
&= e^x \left[\begin{array}{c} \sum_{j=0}^{\left[\frac{n-1}{rk}\right]} \frac{(r+\alpha+n-1)!x^{rkj}(-1)^{rkj}}{(rkj\,;k)!} \\ \left\{ \frac{1}{(n-1-rkj\,;k)!(r+\alpha+rkj)!} + \frac{1}{(n-rkj\,;k)!(r+\alpha+rkj-1)!} \right\} + \frac{x^{rkn}}{(rkn\,;k)!} \end{array} \right] \\
&= e^x \left[\sum_{j=0}^{\left[\frac{n-1}{rk}\right]} \frac{(r+\alpha+n-1)!(-1)^{rkj}}{(n-rkj\,;k)!(r+\alpha+rkj)!} \{r+\alpha+n\} \frac{x^{rkj}}{(rkj\,;k)!} + \frac{x^{rkn}}{(rkn\,;k)!} \right] = e^x \left[\sum_{j=0}^{\left[\frac{n-1}{rk}\right]} \frac{(r+\alpha+n)!(-1)^{rkj}}{(n-rkj\,;k)!(r+\alpha+rkj)!} \frac{x^{rkj}}{(rkj\,;k)!} + \frac{x^{rkn}}{(rkn\,;k)!} \right] \\
&= e^x \left[(r+1+\alpha)_{n,k} \sum_{j=0}^{\left[\frac{n-1}{rk}\right]} \frac{(-1)^{rkj}}{(n-rkj\,;k)!(r+1+\alpha)_{rj,k}} \frac{x^{rkj}}{(rkj\,;k)!} + \frac{x^{rkn}}{(rkn\,;k)!} \right] = e^x (r+1+\alpha)_{n,k} \sum_{j=0}^{\left[\frac{n}{rk}\right]} \frac{(-1)^{rkj}}{(n-rkj\,;k)!(r+1+\alpha)_{rj,k}} \frac{x^{rkj}}{(rkj\,;k)!} = A^{(1+\alpha)}_{r,n,k}(x).
\end{aligned}
\tag{54}
$$

6. K Differential Equation

Theorem 15.
Assume that $\alpha \in \mathbb{R}$ and $n \geq q$. Thus, we reach

$$
xD^2 A^{(\alpha)}_{r,n,k}(x) + (rk+\alpha-3x) DA^{(\alpha)}_{r,n,k}(x) + (2x+n-rk-\alpha) A^{(\alpha)}_{r,n,k}(x) = 0. \tag{55}
$$

Proof.
We have

$$
xD^2 A^{(\alpha)}_{r,n,k}(x) + DA^{(\alpha)}_{r,n,k}(x) = (n+x) DA^{(\alpha)}_{r,n,k}(x) + A^{(\alpha)}_{r,n,k}(x) - (rk+\alpha+n-1) DA^{(\alpha)}_{r,n-1,k}(x). \tag{56}
$$

By using Equation (36), we get

$$
\begin{aligned}
&xD^2 A^{(\alpha)}_{r,n,k}(x) + DA^{(\alpha)}_{r,n,k}(x) = (n+x) DA^{(\alpha)}_{r,n,k}(x) + A^{(\alpha)}_{r,n,k}(x) \\
&- (rk+\alpha+n-1) \left[DA^{(\alpha)}_{r,n,k}(x) - A^{(\alpha)}_{r,n,k}(x) + 2A^{(\alpha)}_{r,n-1,k}(x) \right],
\end{aligned}
\tag{57}
$$

or

$$
xD^2 A^{(\alpha)}_{r,n,k}(x) + (rk+\alpha-x) DA^{(\alpha)}_{r,n,k}(x) = (rk+\alpha+n) A^{(\alpha)}_{r,n,k}(x) - 2(rk+\alpha+n-1) A^{(\alpha)}_{r,n-1,k}(x). \tag{58}
$$

By using Equation (26), we have

$$
\begin{aligned}
xD^2 A^{(\alpha)}_{r,n,k}(x) + (rk+\alpha-x) DA^{(\alpha)}_{r,n,k}(x) &= (rk+\alpha+n) A^{(\alpha)}_{r,n,k}(x) + 2xDA^{(\alpha)}_{r,n,k}(x) \\
&\times\ -2(n+x) DA^{(\alpha)}_{r,n,k}(x),
\end{aligned}
\tag{59}
$$

or

$$
xD^2 A^{(\alpha)}_{r,n,k}(x) + (rk+\alpha-3x) DA^{(\alpha)}_{r,n,k}(x) + (2x+n-rk-\alpha) A^{(\alpha)}_{r,n,k}(x) = 0. \tag{60}
$$

□

7. K Rodrigues Formula

Theorem 16.
Assume that $\alpha \in \mathbb{R}$ and $n, j, k \in \mathbb{Z}^+$. Thus, we reach

$$
A^{(\alpha)}_{r,n,k}(x) = \frac{x^{-(rk-1)-\alpha} e^{2x}}{(n\,;k)!} D_k^n \left(x^{(rk-1)+\alpha+nk} e^{-x} \right). \tag{61}
$$

Proof.
We take into consideration the K extended Laguerre polynomials involving

$r > 2\,_2F_2, r > 2,$

$$A_{r,n,k}^{(\alpha)}(x) = \frac{e^x (rk+\alpha)_{n,k}}{(n;k)!}\,{}_rF_{r,k}\left(\begin{matrix}\left(\frac{-n}{r},k\right),\left(\frac{-n+k}{r},k\right),\cdots,\left(\frac{-n+rk+1}{r},k\right); \\ \left(\frac{\alpha+rk}{r},k\right),\left(\frac{\alpha+rk+1}{r},k\right),\cdots,\left(\frac{\alpha+2rk-1}{r},k\right)\end{matrix};x^r\right). \tag{62}$$

By Theorem (12), we have

$$\begin{aligned} A_{r,n,k}^{(\alpha)}(x) &= \frac{e^x}{(n;k)!}\sum_{j=0}^{[n/rk]}\left[\frac{(n;k)!}{(n-rkj;k)!(rkj;k)!}\right]\frac{(rk+\alpha)_{n,k}x^{rkj}}{(rk+\alpha)_{rj,k}} \\ &= \frac{e^x x^{-(rk-1)-\alpha}}{(n;k)!}\sum_{j=0}^{[n/r]}\left[\frac{(-1)^{rkj}(n;k)!}{(n-rkj)!(rkj;k)!}\right]\frac{(\alpha+rk)_{n,k}x^{rkj+\alpha+(rk-1)}}{(\alpha+rk)_{rj,k}}. \end{aligned} \tag{63}$$

Since $D^{nk-rjk}(x^{n+\alpha+(r-1)}) = (\alpha+rk)_{n,k}x^{rj+\alpha+(r-1)}/(\alpha+rk)_{rj,k}$, we get

$$\begin{aligned} A_{r,n,k}^{(\alpha)}(x) &= \frac{x^{-(r-1)-\alpha}e^{2x}}{(n;k)!}\sum_{j=0}^{[n/rk]}\left[\frac{(n;k)!}{(n-rjk;k)!(rj)!}\right]\left[(-1)^{rkj}e^{-x}\right]\left[D^{nk-rkj}\left(x^{n+\alpha+(rk-1)}\right)\right] \\ &= \frac{x^{-(r-1)-\alpha}e^{2x}}{(n;k)!}\sum_{j=0}^{[n/rk]^n} C_{rjk;k}D^{nk-rkj}\left(x^{n+\alpha+(rk-1)}\right)D^{rkj}\left(e^{-x}\right) \end{aligned} \tag{64}$$

Then, we get our desired result. □

8. Special Properties

Theorem 17.
Suppose that $\alpha, \beta \in \mathbb{R}$ and $n, j, r, k \in \mathbb{Z}^+$. Thus, we acquire

$$A_{r,n,k}^{(\alpha)}(x) = \sum_{j=0}^{[n/rk]}\frac{(\alpha-\beta)_{rkj}A_{r,n-rkj}^{(\beta)}(x)}{(rkj;k)!}. \tag{65}$$

Proof.
From Equation (25),

$$\sum_{n=0}^{\infty}A_{r,n,k}^{(\alpha)}(x)t^n = \frac{1}{(1-t)^{rk+\alpha}}\exp\left(\frac{x-2xt}{1-t}\right). \tag{66}$$

Also, consider

$$\begin{gathered} \frac{1}{(1-t)^{rk+\alpha}}\exp\left(x\left(\frac{1-2t}{1-t}\right)\right) = (1-t)^{-(\alpha-\beta)}(1-t)^{-rk-\beta}\exp\left(x\left(\frac{1-2t}{1-t}\right)\right), \\ \sum_{n=0}^{\infty}A_{r,n,k}^{(\alpha)}(x)t^n = (1-t)^{-(\alpha-\beta)}\sum_{n=0}^{\infty}A_{r,n,k}^{(\beta)}(x)t^n = \sum_{n=0}^{\infty}\frac{(\alpha-\beta)_{rn}t^{rn}}{(rn;k)!}\sum_{n=0}^{\infty}A_{r,n,k}^{(\beta)}(x)t^n = \sum_{n=0}^{\infty}\sum_{j=0}^{\infty}\frac{(\alpha-\beta)_{rkj}t^{rkj}A_{r,n,k}^{(\beta)}(x)t^n}{(rkj;k)!}. \end{gathered} \tag{67}$$

By using Lemma (4) yields

$$\sum_{n=0}^{\infty}A_{r,n,k}^{(\alpha)}(x)t^n = \sum_{n=0}^{\infty}\sum_{j=0}^{[n/rk]}\frac{(\alpha-\beta)_{rkj}t^{rkj}A_{r,n-rkj,k}^{(\beta)}(x)t^{n-rkj}}{(rkj;k)!} = \sum_{n=0}^{\infty}\sum_{j=0}^{[n/rk]}\frac{(\alpha-\beta)_{rkj}A_{r,n-rkj,k}^{(\beta)}(x)t^n}{(rkj;k)!}. \tag{68}$$

Then, we get our result. □

Theorem 18.
If $\alpha \in \mathbb{R}$ and $n, j, k \in \mathbb{Z}^+$, then

$$A_{r,n,k}^{(\alpha+\beta+qk)}(x+y) = \sum_{j=0}^{[n/rk]}A_{r,n-rkj,k}^{(\beta)}(y)A_{r,rkj,k}^{(\alpha)}(x). \tag{69}$$

Proof.
Consider

$$\begin{aligned} &(1-t)^{-rk-\alpha}\exp\left(x\left(\frac{1-2t}{1-t}\right)\right)(1-t)^{-rk-\beta}\exp\left(y\left(\frac{1-2t}{1-t}\right)\right) \\ &= (1-t)^{-rk-(\alpha+\beta+rk)}\exp\left\{(x+y)\left(\frac{1-2t}{1-t}\right)\right\}. \end{aligned} \tag{70}$$

Then, we get

$$\sum_{n=0}^{\infty}A_{r,n,k}^{(\alpha)}(x)t^n\sum_{n=0}^{\infty}A_{r,n,k}^{(\beta)}(y)t^n = \sum_{n=0}^{\infty}A_{r,n,k}^{(\alpha+\beta+q)}(x+y)t^n. \tag{71}$$

By using Lemma (4), we acquire

$$\sum_{n=0}^{\infty} A_{r,n,k}^{(\alpha+\beta+qk)}(x+y)t^n = \sum_{n=0}^{\infty}\sum_{j=0}^{[n/rk]} A_{r,n-rkj,k}^{(\beta)}(y)A_{r,rkj,k}^{(\alpha)}(x)t^n. \quad (72)$$

On comparing the coefficients of t^n, we acquire our result. □

9. Conclusion

We constructed the K extended Laguerre polynomials $\{A_{r,n,k}^{(\alpha)}(x)\}$ relied on the ${}_rF_r$, $r>2$. We acquired K generating functions, K recurrence relations and K Rodrigues formula for these K extended Laguerre polynomials. We will use the integral transformations on the results of K extended Laguerre polynomials in our future works (Table 1). We can also apply Laplace transformation on our results.

Data Availability

No data were used to support this work.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

The authors declare that the study was realized in collaboration with equal responsibility. All authors read and approved the final manuscript.

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