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Existence of Coupled Quasi-solutions of Nonlinear Integro-Differential Equations of Volterra Type in Banach Spaces

Yan Liang[1*]

[1] *Zhengzhou Infant Normal College, Zhengzhou 450000, China.*

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Abstract

We study an initial value problem for a class of integro-differential equations of Volterra type in a real Banach space. Using method of upper and lower solutions and Mönch and Von Harten theorem, we obtain an existence theorem of coupled quasi-solutions, which is an extension of those established by Y. Chen and W. Zhuang in [1].

Keywords: Banach space; measure of noncompactness; lower and upper solutions; normal cone; Quasi-solution.

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1 Introduction and Preliminaries

Let E be a real Banach space with norm $\|\cdot\|$, and let E^* denote the dual of E. Let $K \subset E$ be a cone. By means of K a partial order $\leq$ is defined as $v \leq u$ iff $u - v \in K$. We let $K^* = \{\varphi \in E^* : \varphi(u) \geq 0$ for all $u \in K\}$.

**Corresponding author: E-mail: 306572265@qq.com*

A cone K is said to be normal if there exists a real number $\delta > 0$ such that $0 \leq v \leq u$ implies $\|v\| \leq \delta\|u\|$, where δ is independent of u, v. We shall always assume in this paper that K is a normal cone.

Let β denote the measure of noncompactness of Hausdorff respectively. If F is a subspace of E and $M \subset F$ is bounded then we define

$$\beta_F(M) = \inf\{\varepsilon > 0 : M \subset \bigcup_{i=1}^{n(\varepsilon)} S(z_i^\varepsilon, \varepsilon) \text{ for some } z_i^\varepsilon \in F\}.$$

We have

$$\beta(B) \leq \beta_F(B) \leq 2 \cdot \beta(B) \text{ for } B \subset F \text{ bounded.}$$

For these and further properties we refer to Deimling [2] and Sadovskii [3].

For any $v, w \in C[I, E]$ such that $v(t) \leq w(t)$ on I, where $I = [0, T], T > 0$, we define the conical segment

$$[v, w] = \{u \in C[I, E] : v \leq u \leq w\}.$$

From the definition of $[v, w]$ and the normality of the cone K, we know that $[v, w]$ is a bounded closed convex subset of $C[I, E]$.

In this paper, we consider the following initial value problem for nonlinear integro-differential equation (IVP) in a real Banach space, namely

$$u^{'}(t) = H(t, u(t), (Su)(t)), \quad u(0) = u_0 \tag{1.1}$$

where $(Su)(t) = \int_0^t s(t, \tau)u(\tau)d\tau, s \in C[I \times I, R^+]$ and $s(t, s) \leq s_0$ on $I \times I, s_0 > 0, H \in C[I \times E \times E, E]$. We obtain an existence theorem of coupled quasi-solutions via the method of upper and lower solutions and Mönch and Von Harten theorem. The results of this paper are extensions of those established in [1].

In the proof of our main results the following lemmas are necessary. See [4],[5],[6] for details.

Lemma 1.1[5] **(Mönch and Von Harten theorem)** Let E be a Banach space and β the Hausdorff measure of noncompactness on E. Let $\{x_n\}_{n\geq 1}$ be a sequence of continuously differentiable functions from $J = [a, b]$ to E such that there is some $\mu \in L^1(a, b)$ with $\|x_n(t)\| \leq \mu(t)$ and $\|x_n^{'}(t)\| \leq \mu(t)$ on J. Let $\psi(t) = \beta(\{x_n(t)\}_{n\geq 1})$. Then $\psi(t)$ is absolutely continuous on J and

$$\psi^{'}(t) \leq 2\beta(\{x_n^{'}(t)\}_{n\geq 1}) \quad a.e. \text{ on } J.$$

Lemma 1.2[5] Let E be a separable Banach space and β the Hausdorff measure of noncompactness on E. Let $\{x_n\}_{n\geq 1}$ be a sequence of continuous functions from $J = [a, b]$ to E such that there is some $\mu \in L^1(a, b)$ with $\|x_n(t)\| \leq \mu(t)$ on J. Let $\psi(t) = \beta(\{x_n(t)\}_{n\geq 1})$. Then $\psi(t)$ is integrable on J and

$$\beta(\{\int_a^b x_n(s)ds\}_{n\geq 1}) \leq \int_a^b \psi(s)ds.$$

Lemma 1.3[6] Let $y(t) \in C[I, R], y(0) \leq 0$, and satisfy

$$y^{'}(t) \leq -My(t) - N\int_0^t s(t, \tau)y(\tau)d\tau$$

where $M > 0, N \geq 0, s \in C[I \times I, R^+], s(t, \tau) \leq s_0$ for $(t, \tau) \in I \times I$. Suppose further that $Ns_0T(exp(MT) - 1) \leq M$. Then $y(t) \leq 0$ for all $t \in I$.

To define appropriate classes of upper and lower solutions of (1.1) , we shall suppose that H admits a decomposition of the form

$$H(t,u,Su) = H_0(t,u,Su) + H_1(t,u,Su) + H_2(t,u,Su),$$

where $H_0, H_1, H_2 \in C[I \times E \times E, E]$.

Definition 1.1 Let $v_0, w_0 \in C^1[I,E]$. Then v_0, w_0 are said to be coupled lower and upper quasi-solutions of (1.1) if

$$\begin{cases} v_0^{'} - H_0(t,v_0,Sv_0) - H_1(t,v_0,Sv_0) - H_2(t,w_0,Sw_0) \le 0, \quad v_0(0) \le u_0, \\ w_0^{'} - H_0(t,w_0,Sw_0) - H_1(t,w_0,Sw_0) - H_2(t,v_0,Sv_0) \ge 0, \quad w_0(0) \ge u_0. \end{cases} \tag{1.2}$$

If in (1.2), equalities hold, then v_0, w_0 are said to be coupled quasi-solutions of (1.1). Clearly one can define, based on definition 1.1, coupled maximal and minimal quasi-solutions of (1.1). We also need a stronger form of coupled upper and lower quasi-solutions of (1.1).

Definition 1.2 Let $v_0, w_0 \in C^1[I,E]$ be such that $v_0(t) \le w_0(t)$ on I. Then v_0, w_0 are said to be strongly coupled lower and upper quasi-solutions of (1.1) if there exist constants $M > 0, N \ge 0$ such that

$$\begin{cases} v_0^{'} \le H_0(t,\sigma,S\sigma) + H_1(t,v_0,Sv_0) + H_2(t,w_0,Sw_0) - M(v_0-\sigma) - N(Sv_0 - S\sigma) \\ w_0^{'} \ge H_0(t,\sigma,S\sigma) + H_1(t,w_0,Sw_0) + H_2(t,v_0,Sv_0) - M(w_0-\sigma) - N(Sw_0 - S\sigma) \end{cases} \tag{1.3}$$

for all $\sigma \in [v_0, w_0]$.

We list for convenience the following assumptions and suppose that $v_0, w_0 \in C^1[I,E]$ such that $v_0(t) \le w_0(t)$ on I and $Ns_0T(exp(MT) - 1) \le M$.

(A_1) For any bounded set $B \subset [v_0, w_0]$,

$\beta(\{H_0(t,x,Sx) + H_1(t,x,Sx) : x \in B\}) \le L\beta(B(t))$,

$\beta(\{H_2(t,x,Sx) : x \in B\}) \le L\beta(B(t))$,

where $L > 0, B(t) = \{x(t) : x \in B\}$.

($A_1^{'}$) For any $u, v \in [v_0, w_0]$,

$\|(H_0 + H_1)(t,u,Su) - (H_0 + H_1)(t,v,Sv)\| \le L\|u(t) - v(t)\|$,

$\|H_2(t,u,Su) - H_2(t,v,Sv)\| \le L\|u(t) - v(t)\|$,

where $L > 0$.

(A_2) $H_1(t,x,Sx)$ is nondecreasing in x and $H_2(t,x,Sx)$ is nonincreasing in x relatively to the normal cone K.

Note Clearly, ($A_1^{'}$) implies (A_1).

2 Main Results

Our main aim in this paper is to prove the following theorem.

Theorem 2.1 Assume that the cone K is normal and assumptions ($A_1^{'}$),(A_2) and (1.3) are satisfied. Then there exists a unique solution $u(t)$ of (1.1) on I such that $u \in [v_0, w_0]$, provided $v_0(0) \le u_0 \le w_0(0)$.

In our paper, $H = H_0 + H_1 + H_2$, where H_0, H_1, H_2 satisfy different conditions respectively. The papers in [7-12] were concerned with single H , their conditons of measure of noncompactness were relatively strong.

The proof of the above theorem will be completed by a series lemmas.

First of all, we consider the following linear initial value problem (LIVP):

$$u^{'} = \overline{H}(t,u,Su), \quad u(0) = u_0 \tag{2.1}$$

where $\overline{H}(t,u,Su) = -Mu - N(Su) + H_0(t,\eta_1,S\eta_1) + H_1(t,\eta_1,S\eta_1) + H_2(t,\eta_2,S\eta_2) + M\eta_1 + N(S\eta_1)$, and $\eta_1, \eta_2 \in [v_0, w_0]$. Then we have

Lemma 2.1 For any $\eta_1, \eta_2 \in [v_0, w_0]$, there exists a unique solution $u(t)$ on I of (2.1).

The proof of this lemma is similar to a corresponding result given in [6] with minor modifications.

For any $\eta_1, \eta_2 \in [v_0, w_0]$, we define the mapping A by $A[\eta_1, \eta_2] = u$, where $u = u(t)$ is the unique solution of (2.1) on I corresponding to η_1, η_2. Then we have the following

Lemma 2.2 Suppose that assumptions (A$_2$) and (1.3) hold. Then A maps $[v_0, w_0] \times [v_0, w_0]$ into $[v_0, w_0]$.

Proof. Let $\eta_1, \eta_2 \in [v_0, w_0]$ and let $u = A[\eta_1, \eta_2]$. For any $\phi \in K^*$, set $p(t) = \phi(v_0(t) - u(t))$ and note that $p(0) = \phi(v_0(0) - u_0) \le 0$. Then for all $\sigma \in [v_0, w_0]$, we have

$$\begin{aligned} p^{'}(t) &= \phi(v_0^{'}(t) - u^{'}(t)) \\ &\le \phi(H_0(t, \sigma, S\sigma) + H_1(t, v_0, Sv_0) + H_2(t, w_0, Sw_0) - M(v_0 - \sigma) - N(Sv_0 - S\sigma) \\ &\quad -H_0(t, \eta_1, S\eta_1) - H_1(t, \eta_1, S\eta_1) - H_2(t, \eta_2, S\eta_2) + M(u - \eta_1) + N(Su - S\eta_1)). \end{aligned}$$

Choosing $\sigma = \eta_1$, then we get

$$\begin{aligned} p^{'}(t) &\le \phi((H_1(t, v_0, Sv_0) - H_1(t, \eta_1, S\eta_1)) + (H_2(t, w_0, Sw_0) - H_2(t, \eta_2, S\eta_2))) \\ &\quad -\phi(M(v_0 - u) - N(Sv_0 - Su)) \\ &\le -Mp(t) - N\int_0^t s(t, \tau)p(\tau)d\tau, \end{aligned}$$

which implies $p(t) \le 0$ by Lemma 1.3. This proves $v_0(t) \le u(t)$ on I since $\phi \in K^*$ is arbitrary.

A similar argument yields that $u(t) \le w_0(t)$ on I. Since $\eta_1, \eta_2 \in [v_0, w_0]$ are arbitrary, the proof is complete. □

In view of Lemma 2.2, we can define the sequences $\{v_n\}, \{w_n\}$ as follows:

$$v_{n+1} = A[v_n, w_n], w_{n+1} = A[w_n, v_n] \text{ and } v_n, w_n \in [v_0, w_0], n = 0, 1, 2, \cdots.$$

We now prove the following lemma.

Lemma 2.3 Suppose that assumptions (A$_1$), (A$_2$) and (1.3) hold. Then the sequences $\{v_n\}, \{w_n\}$ are uniformly bounded, equicontinuous and relatively compact on I.

Proof. Since the cone K is normal and $v_n, w_n \in [v_0, w_0]$ for all $t \in I$, it follows that $\{v_n\}, \{w_n\}$ are uniformly bounded on I.

By assumption (A$_1$), $H_0 + H_1$ and H_2 map a bounded set into a bounded set. Noting that

$$\begin{aligned} v_n^{'}(t) &= -Mv_n - N(Sv_n) + H_0(t, v_{n-1}, Sv_{n-1}) + H_1(t, v_{n-1}, Sv_{n-1}) \\ &\quad +H_2(t, w_{n-1}, Sw_{n-1}) + Mv_{n-1} + N(Sv_{n-1}), \end{aligned} \tag{2.2}$$

so there exists a constant $M_0 > 0$ such that

$$\|v_n^{'}(t)\| \le M_0, \quad t \in I, \quad n = 1, 2, \cdots.$$

Therefore, by the mean value theorem (see theorem 1.3.2 in [13]), we have

$$\|v_n(t_1) - v_n(t_2)\| \le M_0|t_1 - t_2|, \quad t_1, t_2 \in I.$$

This implies that $\{v_n(t)\}$ is equicontinuous on I. A similar argument shows that $\{w_n(t)\}$ is equicontinuous on I.

Let $\varphi(t) = \beta(\{v_n(t) : n \ge 0\}), \quad \psi(t) = \beta(\{w_n(t) : n \ge 0\})$. Clearly $\{v_n(t)\}, \{w_n(t)\}$ satisfy the conditions of Lemma 1.2. Therefore

$$\varphi^{'}(t) \le 2\beta(\{v_n^{'}(t) : n \ge 0\}), \quad \psi^{'}(t) \le 2\beta(\{w_n^{'}(t) : n \ge 0\}) \quad \text{a.e. on } I. \tag{2.3}$$

Let $E_1 = \overline{span}\{v_n(t), w_n(t) : n \ge 0, t \in I \cap Q\}$, where Q is the set of rational numbers. By assumption (A$_1$), we have

$$\beta(\{H_0(t, v_{n-1}, Sv_{n-1}) + H_1(t, v_{n-1}, Sv_{n-1}) : n \ge 1\}) \le L\beta(\{v_{n-1}(t) : n \ge 1\}) = L\varphi(t), \tag{2.4}$$

$$\beta(\{H_2(t, w_{n-1}, Sw_{n-1}) : n \geq 1\}) \leq L\beta(\{w_{n-1}(t) : n \geq 1\}) = L\psi(t). \tag{2.5}$$

By the properties of Hausdorff's measure of noncompactness and Lemma 1.2, we obtain

$$\begin{aligned}
\beta(\{(Sv_{n-1})(t) : n \geq 1\}) &\leq \beta_{E_1}(\{(Sv_{n-1}) : n \geq 1\}) = \beta_{E_1}(\{\int_0^t s(t,\tau)v_{n-1}(\tau)d\tau : n \geq 1\}) \\
&\leq \int_0^t \beta_{E_1}(\{s(t,\tau)v_{n-1}(\tau) : n \geq 1\})d\tau = \int_0^t s(t,\tau)\beta_{E_1}(\{v_{n-1}(\tau) : n \geq 1\})d\tau \\
&\leq 2\int_0^t s(t,\tau)\beta(\{v_{n-1}(\tau) : n \geq 1\})d\tau \leq 2\int_0^t s_0\varphi(\tau)d\tau.
\end{aligned} \tag{2.6}$$

By (2.2)-(2.6), we have

$$\begin{aligned}
\varphi^{'}(t) &\leq 2(2M\varphi(t) + 2N \cdot 2s_0 \int_0^t \varphi(\tau)d\tau + L\varphi(t) + L\psi(t)) \\
&= 2(2M + L)\varphi(t) + 2L\psi(t) + 8Ns_0 \int_0^t \varphi(\tau)d\tau \quad \text{a.e. on } I.
\end{aligned}$$

A similar argument yields that

$$\psi^{'}(t) \leq 2(2M + L)\psi(t) + 2L\varphi(t) + 8Ns_0 \int_0^t \psi(\tau)d\tau \quad \text{a.e. on } I.$$

Set $m(t) = \varphi(t) + \psi(t)$, then

$$m'(t) \leq 2(2M + L)m(t) + 2Lm(t) + 8Ns_0 \int_0^t m(\tau)d\tau, \quad \text{a.e. on } I,$$

that is,

$$m'(t) \leq 4(M + L)m(t) + 8Ns_0 \int_0^t m(\tau)d\tau \quad \text{a.e. on } I.$$

Noting that $m(0) = 0$, we have

$$\begin{aligned}
m(t) &\leq 4(M + L)\int_0^t m(\tau)d\tau + 8Ns_0 \int_0^t \int_0^\tau m(\xi)d\xi d\tau \\
&\leq 4(M + L)\int_0^t m(\tau)d\tau + 8Ns_0 \int_0^t \int_0^t m(\xi)d\xi d\tau \\
&\leq 4(M + L)\int_0^t m(\tau)d\tau + 8Ns_0T \int_0^t m(\xi)d\xi \\
&= 4(M + L + 2Ns_0T)\int_0^t m(\tau)d\tau.
\end{aligned}$$

Consequently, $m(t) \leq m(0)exp(4(M + L + 2Ns_0T)) = 0$, we thus obtain $\varphi(t) = \psi(t) = 0, t \in I$. Hence, by Ascoli-Arzela theorem, the sequences $\{v_n\}, \{w_n\}$ are relatively compact in $C[I, E]$. The proof of Lemma 2.3 is complete. □

Lemma 2.4 Suppose that assumptions ($A_1^{'}$), (A_2) and (1.3) hold. Then we have $p(t) \equiv 0$ on I for either

$$p(t) = \overline{\lim_{n\to\infty}}\|v_n(t) - v_{n-1}(t)\|$$

or

$$p(t) = \overline{\lim_{n\to\infty}}\|w_n(t) - w_{n-1}(t)\|.$$

Proof. Let $m_1(t) = \overline{\lim_{n\to\infty}}\|v_n(t) - v_{n-1}(t)\|$, $m_2(t) = \overline{\lim_{n\to\infty}}\|w_n(t) - w_{n-1}(t)\|$, $m(t) = m_1(t) + m_2(t)$. In the following we will prove that $m(t) \equiv 0$ on I.

By the proof of Lemma 2.3, there exists a constant $M_0 > 0$ such that

$$\|v_n^{'}(t)\| \leq M_0, \quad \|w_n^{'}(t)\| \leq M_0, \quad \forall t \in I.$$

Therefore, for all $t_1, t_2 \in I$, we have

$$\begin{aligned}&|\|v_n(t_1) - v_{n-1}(t_1)\| - \|v_n(t_2) - v_{n-1}(t_2)\|| \\ &\leq \|v_n(t_1) - v_n(t_2)\| + \|v_{n-1}(t_1) - v_{n-1}(t_2)\| \\ &\leq 2M_0|t_1 - t_2|,\end{aligned}$$

thus

$$\begin{aligned}\|v_n(t_1) - v_{n-1}(t_1)\| &\leq \|v_n(t_2) - v_{n-1}(t_2)\| + 2M_0|t_1 - t_2|, \\ \|v_n(t_2) - v_{n-1}(t_2)\| &\leq \|v_n(t_1) - v_{n-1}(t_1)\| + 2M_0|t_1 - t_2|.\end{aligned}$$

Taking the limit, we obtain

$$m_1(t_1) \leq m_1(t_2) + 2M_0|t_1 - t_2|, \quad m_1(t_2) \leq m_1(t_1) + 2M_0|t_1 - t_2|$$

and so

$$|m_1(t_1) - m_1(t_2)| \leq 2M_0|t_1 - t_2|,$$

which proves that $m_1(t)$ is continuous on I. A similar argument yields that $m_2(t)$ is also continuous on I.

Now ($A_1^{'}$) yields

$$\begin{aligned}\|v_{n+1}(t) - v_n(t)\| \leq & \int_0^t [\|H_0(\tau, v_n(\tau), (Sv_n)(\tau)) + H_1(\tau, v_n(\tau), (Sv_n)(\tau)) \\ & -H_0(\tau, v_{n-1}(\tau), (Sv_{n-1})(\tau)) - H_1(\tau, v_{n-1}(\tau), (Sv_{n-1})(\tau))\| \\ & +\|H_2(\tau, w_n(\tau), (Sw_n)(\tau)) - H_2(\tau, w_{n-1}(\tau), (Sw_{n-1})(\tau))\| + M\|v_{n+1} - v_n\| \\ & +N\|Sv_{n+1} - Sv_n\| + M\|v_n - v_{n-1}\| + N\|Sv_n - Sv_{n-1}\|]d\tau \\ \leq & \int_0^t [(L+M)\|v_n - v_{n-1}\| + L\|w_n - w_{n-1}\| + M\|v_{n+1} - v_n\| \\ & +N\|Sv_{n+1} - Sv_n\| + N\|Sv_n - Sv_{n-1}\|]d\tau \\ \leq & \int_0^t [(L+M)\|v_n - v_{n-1}\| + L\|w_n - w_{n-1}\| \\ & +(M + Ns_0T)\|v_{n+1} - v_n\| + Ns_0T\|v_n - v_{n-1}\|]d\tau \\ = & \int_0^t [(L+M+Ns_0T)\|v_n - v_{n-1}\| + L\|w_n - w_{n-1}\| \\ & +(M + Ns_0T)\|v_{n+1} - v_n\|]d\tau.\end{aligned}$$

By Fatou lemma, taking limit, we have

$$\begin{aligned}m_1(t) &\leq \int_0^t [(L+M+Ns_0T)m_1(\tau) + Lm_2(\tau) + (M+Ns_0T)m_1(\tau)]d\tau \\ &= \int_0^t [(L+2M+2Ns_0T)m_1(\tau) + Lm_2(\tau)]d\tau.\end{aligned}$$

Similarly, we can obtain

$$m_2(t) \leq \int_0^t [(L+2M+2Ns_0T)m_2(\tau) + Lm_1(\tau)]d\tau.$$

Therefore,

$$m(t) = m_1(t) + m_2(t) \leq 2\int_0^t (L+M+Ns_0T)m(\tau)d\tau.$$

Notice that $m(0) = 0$, so $m(t) \equiv 0$ on I. The proof of Lemma 2.4 is complete. □

Proof of theorem 2.1

By Lemma 2.3, the sequences $\{v_n\}, \{w_n\}$ have uniformly convergent subsequences $\{v_{n_k}\}, \{w_{n_k}\}$. We let $\lim_{k\to\infty} v_{n_k} = v$, $\lim_{k\to\infty} w_{n_k} = w$ and notice that

$$\begin{aligned}v_{n_k}(t) = u_0 + & \int_0^t [-Mv_{n_k}(\tau) - N(Sv_{n_k})(\tau) + H_0(\tau, v_{n_k-1}(\tau), (Sv_{n_k-1})(\tau)) \\ & +H_1(\tau, v_{n_k-1}(\tau), (Sv_{n_k-1})(\tau)) + H_2(\tau, w_{n_k-1}(\tau), (Sw_{n_k-1})(\tau)) + Mv_{n_k-1}(\tau) + N(Sv_{n_k-1})(\tau)]d\tau,\end{aligned}$$

and Lemma 2.4 implies

$$\lim_{k\to\infty} v_{n_k-1} = \lim_{k\to\infty} v_{n_k} = v, \lim_{k\to\infty} w_{n_k-1} = \lim_{k\to\infty} w_{n_k} = w \text{ uniformly on } I.$$

Therefore, leting $k \to \infty$, we get

$$v(t) = u_0 + \int_0^t [H_0(\tau, v(\tau), (Sv)(\tau)) + H_1(\tau, v(\tau), (Sv)(\tau)) + H_2(\tau, w(\tau), (Sw)(\tau))]d\tau.$$

Similarly, we have

$$w(t) = u_0 + \int_0^t [H_0(\tau, w(\tau), (Sw)(\tau)) + H_1(\tau, w(\tau), (Sw)(\tau)) + H_2(\tau, v(\tau), (Sv)(\tau))]d\tau.$$

Thus v, w are the coupled quasi-solutions of (1.1). By (A_1'), we obtain

$$\|v(t) - w(t)\| \le 2L \int_0^t \|v(\tau) - w(\tau)\| d\tau.$$

It then follows that $\|v(t) - w(t)\| \equiv 0$ on I since $\|v(0) - w(0)\| = 0$. That is, $v = w$ is a solution of (1.1). It is easy to prove that the solution of (1.1) is unique by (A_1'). The proof of theorem 2.1 is therefore complete. □

Remark When $H_1 = H_2 = 0$, theorem 2.1 in this paper is just theorem 2.1 in [1].

3 Conclusions

In this paper, we obtain an existence theorem of coupled quasi-solutions for a class of integro-differential equations of Volterra type in a real Banach space by using method of upper and lower solutions and Mönch and Von Harten theorem.

Competing Interests

The author declares that no competing interests exist.

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