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Controlled Variational Iteration Method for Bratu Equation Arising in Electro-Spun Organic Nanofibers Elaboration

Mohamed M. Mousa[1,2*]

[1]Department of Basic Science, Faculty of Engineering at Benha, Benha University, 13512, Egypt.
[2]Department of Mathematics, College of Sciences and Human Studies at Howtat Sudair, Majmaah University, 11982, Saudi Arabia.

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Abstract

In this paper, an effective formulation of the variational iteration method is suggested for solving Bratu equation arising in electro-spinning. The suggested formulation depends on embedding a nonzero auxiliary parameter that controls the solution convergence region. An alternative formulation of the Bratu equation is suggested as well. The proposed formula eliminates the complexity that appears when solving using the standard variational iteration algorithms illustrated in [1,2] without approximating the exponential term. A suitable choice of the auxiliary parameter results in an accurate approximation compared with the approximation of the standard variational iteration method.

Keywords: Controlled variational iteration method, analytical solution, Bratu equation, electro-spinning.

**Corresponding author: dr.eng.mmmm@gmail.com;*

1 Introduction

Electro-spinning is a process for elaborating Nano fibers by driving a fluidified polymer through a spinneret with the aid of an electric field. In literature, there are many useful models that describe electro-spinning process e.g., [3–5]. Among which Wan–Guo–Pan model is the most famous model driven in [1, 6], which reads

$$\frac{d^2v}{dz^2}+\lambda\, e^{v} = 0. \tag{1}$$

subjected to boundary conditions $v(a)=c_1;\ v(b)=c_2$, where $a,\ b,\ c_1,\ c_2$ are constants.

This famous equation is called by Bratu equation [7]. Here, $v=-6\ln u$, where u is the jet velocity in the spinning process.

The authors in [1,2] suggested the following variational iteration formula by the so-called enhanced variational iteration method in [1] which is the same that suggested in [2] using the standard variational iteration method (VIM) firstly proposed in [8].

$$v_{n+1}(z)=v_{n+1}(z)+\int_0^z \{(s-z)[\frac{d^2v_n(s)}{ds^2}+\lambda\, e^{v_n(s)}]\}\,ds, \qquad n=0,\ 1,\ 2,... \tag{2}$$

A suitable choice of the initial guess $v_0(z)$ that based on the initial conditions results in an accurate solution as n increases. The standard variational iteration formula shown in Eq. (2) that used by other authors [9,10] for solving Bratu equation has two mainly drawbacks. The first drawback is related to the Bratu equation that causes a complexity in the formula because of the difficulty of the integration that appears after the first iteration due to the presence of e^{v} term. All the authors that used VIM for solving Bratu equation overcame this problem by approximating the term e^{v} using Maclaurin series i.e. $e^{v}=1+\frac{v}{1!}+\frac{v^2}{2!}+\frac{v^3}{3!}+\ldots$. This approximation is used in [1] for solving the considered example but not mentioned by the authors. The second drawback is related to the VIM itself which is summarized in the lack the method of a way to improve and control the solution accuracy. It's worth to note that the model of Eq. (1) can be effectively solved by many other methods. Recall, for example, the homotopy analysis method as in [11].

In this paper, we suggest a new formulation of Bratu equation to eliminate the first draw back. A controlled variational iteration method (CVIM) will be considered to eliminate the second draw back as well.

2 New Formulation of Bratu Equation

Multiplying Eq. (1) by $\frac{dv}{dz}$ and integrating from a to z, it can be re-formulated as

$$\frac{d^2v}{dz^2}-\frac{1}{2}\left(\frac{dv}{dz}\right)^2+\frac{1}{2}\alpha^2+\lambda\, e^{c_1}=0. \tag{3}$$

where $\alpha=\left.\frac{dv}{dz}\right|_{z=a}$ subjected to new initial conditions $v(a)=c_1;\ v'(a)=\alpha.$ If the considered problem is an initial value one, then the value of α is obtained from the given data. While in case of the considered boundary value problem, α can be easily obtained analytically or graphically by imposing the condition $v(b)=c_2$ in the solution and solving the obtained equation for α. It can be seen in Eq. (3) that the term e^v is eliminated. Moreover, it is known that the VIM is more efficient in solving initial value problems instead of solving boundary value problems.

3 Controlled Variational Iteration Method

In order to increase and control the convergence region of the VIM series solution, we propose embedding a nonzero auxiliary parameter χ in the standard variational iteration algorithm which will play an important role in improving and control the solution convergence region and accuracy. So, the controlled variational iteration algorithm can be read as:

$$v_{n+1}(z,\chi)=v_{n+1}(z,\chi)+\chi\int_0^z\{(s-z)[\frac{d^2v_n(s,\chi)}{ds^2}-\frac{1}{2}\left(\frac{dv_n(s,\chi)}{ds}\right)^2+\frac{1}{2}\alpha^2+\lambda\, e^{c_1}]\}\,ds. \tag{4}$$

The suitable initial guess $v_0(z,\chi)=c_1+\alpha\, z$ results in more accurate solution as n increases i.e., $v(z)=\lim_{n\to\infty}v_n(z)$. A relatively accurate solution can be obtained by one or few iterations. From Eq. (4), it can be noticed that the CVIM will provide a family of solutions, dependent upon the convergence control parameter χ. If $\chi=1$, the CVIM is exactly the standard VIM. To obtain an accurate approximation to the considered problem, an optimal value of χ must be found. Firstly, the valid region of χ that achieves the solution convergence can be obtained via the χ curve as follows. If $a_0\in[a,\ b]$, then $v_{n+1}(a_0,\chi)$ is a function of χ and the plot of $v_{n+1}(a_0,\chi)$ versus χ contains a horizontal line segment which corresponds to the valid region of χ. This is due to that all convergent series given by different values of the auxiliary parameter χ converge to its exact value. So, if the solution is unique, then all of these series solutions converge to the same value and therefore there exists a horizontal line segment in the curve, all of these possible values of χ construct a set R_χ for the convergence-control parameter. Secondly, a more accurate approximation can be obtained by assigning χ an optimal value. The optimal value χ^* of the control parameter can be estimated by minimizing the averaged absolute residual error $\Delta_n(\chi)$ defined as:

$$\Delta_n(\chi) = \frac{1}{k}\sum_{i=0}^{k}\left|\left(\frac{d^2 v_{n+1}(z,\chi)}{dz^2} - \frac{1}{2}\left(\frac{dv(z,\chi)}{dz}\right)^2 + \frac{1}{2}\alpha^2 + \lambda e^{c_1}\right)\bigg|_{z=i\Delta z}\right|, \tag{5}$$

Where k(integer) $\geq (b-a)$ and $k = \dfrac{b-a}{\Delta z}$, where Δz is the resolution. The low value of Δz leads to obtain a more accurate value of χ^* and hence the solution accuracy will be increased. It's worth to note that the considered formulation is firstly proposed by Turkyilmazoglu in [12].

4 Numerical Examples

To demonstrate the reliability and efficiency of the proposed CVIM, two initial/boundary value problem of Bratu equation are considered. The results of CVIM and standard VIM are compared with the exact solutions at the same number of iterations to demonstrate the accuracy of the considered formulation over the classical VIM.

4.1 Consider the BratuInitial Value Problem [1]

$$\begin{aligned} &\frac{d^2 v}{dz^2} - 2e^v = 0, \quad 0 < z < 1, \\ &v(0) = 0, \ v'(0) = 0. \end{aligned} \tag{6}$$

whose exact solution is $v(z) = -2\ln(\cos z)$.

Using the new formulation of the Bratu equation and CVIM algorithm with $\lambda = -2$, $a = 0$, $b = 1$, $c_1 = \alpha = 0$, $n = 0,\ 1,\ 2$ and $\Delta z = 0.01$. Two and three iterations approximate solutions $v_2(z,\chi)$ and $v_3(z,\chi)$, can be easily obtained as

$$v_2(z,\chi) = \frac{1}{6}\chi^3 z^4 + \left(2\chi - \chi^2\right)z^2. \tag{7}$$

$$v_3(z,\chi) = \frac{1}{252}\chi^7 z^8 + \left(-\frac{2}{45}\chi^6 + \frac{4}{45}\chi^5\right)z^6 + \left(\frac{1}{6}\chi^5 + \frac{5}{6}\chi^3 - \frac{5}{6}\chi^4\right)z^4 + \left(\chi^3 + 3\chi - 3\chi^2\right)z^2. \tag{8}$$

It's noticeably that the new formulation of the Bratu equation, contributed a lot in making the integration and calculations easy.

To find the valid region R_χ for the control parameter χ, the χ curves at $z = 0.4,\ 0.6,\ 0.8$ are drawn in Fig. 1 for the three iterations approximate solution $v_3(z,\chi)$, which clearly indicates that the valid region of χ is about $0.8 \le \chi \le 1.4$ or $R_\chi = [0.8,\ 1.4]$ in which a horizontal line segments appear.

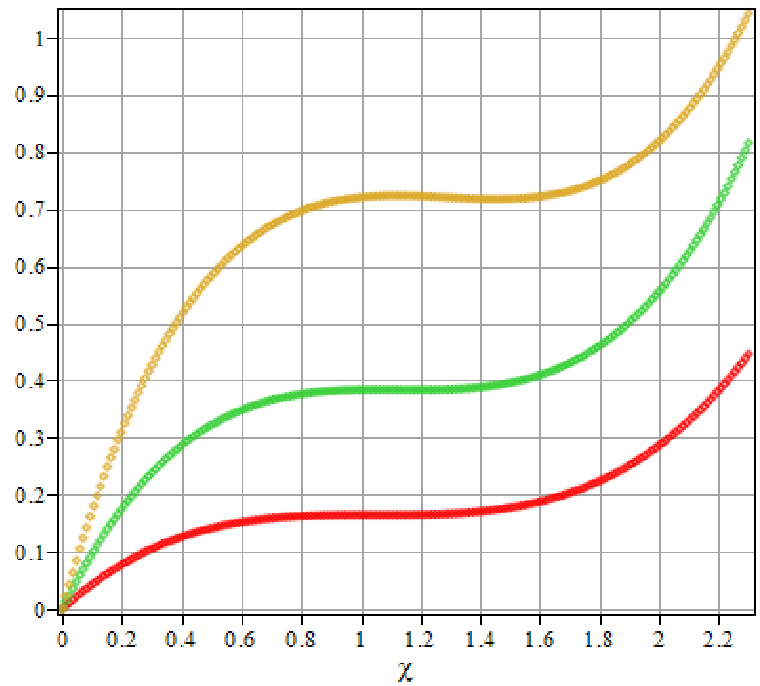
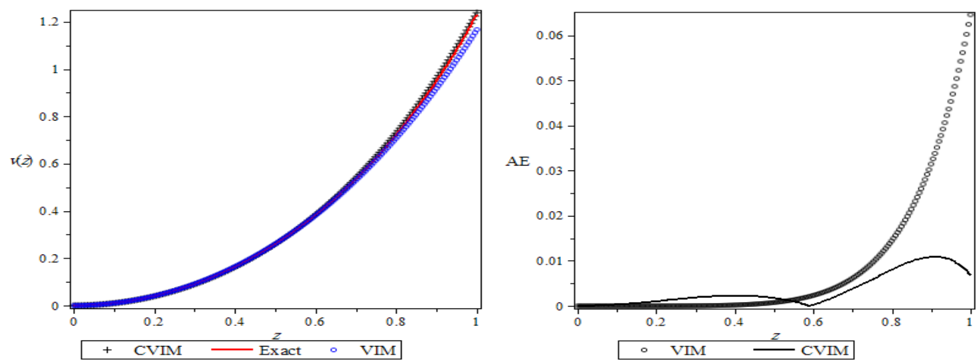


Fig. 1. χ **curves of** $v_3(z,\chi)$ **at** $z = 0.4,\ 0.6,\ 0.8$

Using the low order solution $v_2(z,\chi)$ and minimizing Eq. (5) in the valid region of χ, one find that the optimal value of convergence-control parameter is $\chi^* = 1.17$, at which $\Delta_1(\chi^*) = 0.2065$, while the absolute residual error in case of standard VIM, $\Delta_1(1) = 0.306246$. To illustrate the increase of the accuracy of the CVIM over the standard VIM, Fig. 2 shows a comparison between standard VIM and CVIM solutions along with the exact solution and absolute error (AE) with respect to the exact solution. We use the low order approximation $v_2(z,\chi)$ in this comparison where $\chi = 1.17$ for CVIM solution and $\chi = 1$ for the VIM solution. From Fig. 2, it's clear that the proposed algorithm is more accurate than the standard one. The accuracy of the solution can be increased if we used the three iterations approximate solution $v_3(z,\chi)$.



(a) **(b)**

Fig. 2. A comparison between standard VIM and CVIM results, (a) solutions and (b) absolute error

4.2 Consider the Bratu Boundary Value Problem [13,14]

$$\begin{aligned} &\frac{d^2 v}{dz^2} + \lambda e^{v} = 0, \quad 0 < z < 1, \\ &v(0) = 0, \; v(1) = 0. \end{aligned} \tag{9}$$

whose exact solution is $v(z) = -2\ln\left(\dfrac{\cosh\left(\left(z-\frac{1}{2}\right)\frac{\theta}{2}\right)}{\cosh\left(\frac{\theta}{4}\right)}\right)$, for $\lambda > 0$,

where θ is a solution of the equation $\theta = \sqrt{2\lambda}\,\cosh\left(\dfrac{\theta}{4}\right)$. This Bratu problem possess two solutions, one solution, or no solution provided that $\lambda < \lambda_c$, $\lambda < \lambda_c$, or $\lambda > \lambda_c$, respectively, where the critical value λ_c given by $\lambda_c = 3.513830719$ [14].

Using the new formulation of the Bratu equation and CVIM algorithm with, for example, $\lambda = 3$, $a = 0$, $b = 1$, $c_1 = c_2 = 0$, $n = 0, 1, \ldots, 5$ and $\Delta z = 0.01$. Two iterations approximate solution $v_2(z, \chi)$ and, can be easily obtained as:

$$v_2(z,\chi) = \frac{3}{8}\chi^3 z^4 - \frac{1}{2}\alpha\,\chi^2 z^3 + \left(-3\chi + \frac{3}{2}\chi^2\right) z^2 + \alpha\, z. \tag{10}$$

In order to obtain more accurate solution, more iteration must be calculated. We used the six iterations approximate solution $v_7(z, \chi, \alpha)$ to find the values of the unknown parameter α and the optimal value of the control parameter χ. Firstly, we find a relation between α and χ using the condition $v(1) = 0$ and then plot this relation implicitly i.e., plotting the relation $v_7(1, \chi, \alpha) = 0$ implicitly. The relation between α and χ is plotted in Fig. 3.

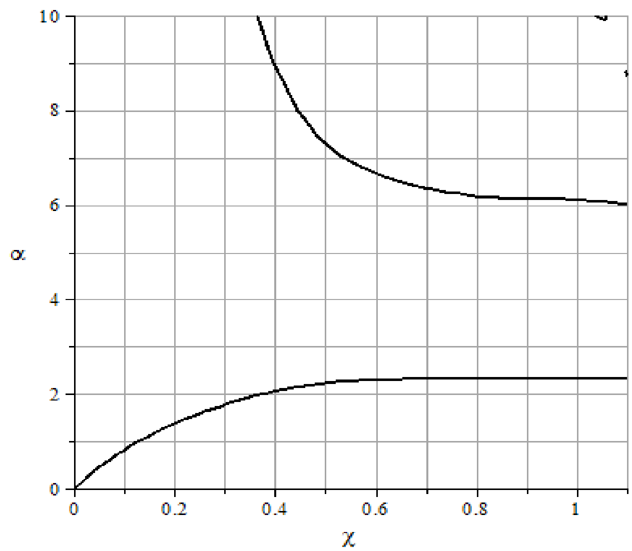


Fig. 3. Plot of α as a function of χ of the relation $v_7(1, \chi, \alpha) = 0$

Form Fig. 3, two values of α are clear (two line segments at which α is approximately constant). The valid region of χ for lower branch solution when $\alpha \approx 2.32$ and for upper branch solution when $\alpha \approx 6.11$ can be chosen as $R_\chi = [0.6,\ 1.1]$. It is clear that, that this problem possess two solutions according to the value of α.

Using the solution $v_7(z, \chi, \alpha)$ and minimizing Eq. (5) in the valid region of χ, one can find that the optimal value of convergence-control parameter is $\chi^* = 0.74$, at which $\Delta 6(\chi^*) = 0.0091$ for the lower branch solution and $\Delta 6(\chi^*) = 0.8012$ for the upper branch solution. The absolute residual error in case of standard VIM, $\Delta_6(1) = 150.617$ when $\alpha \approx 2.32$ and $\Delta_6(1) = 1.666 \times 10^8$ in case of $\alpha \approx 6.11$. To demonstrate the accuracy of the CVIM, Fig. 4 shows a plot of the absolute errors of the CVIM solutions with respect to the upper and lower branch exact solutions. Moreover, Fig. 5 shows a comparison between CVIM solutions and standard VIM solutions along with the exact solutions. It's worth to notice that the lower branch solution is occurred when $\alpha \approx 2.32$ while the upper branch solution is occurred when $\alpha \approx 6.11$. From Figs. 4 and 5, it can be concluded that the proposed CVIM is more accurate and efficient when compared with the standard VIM.

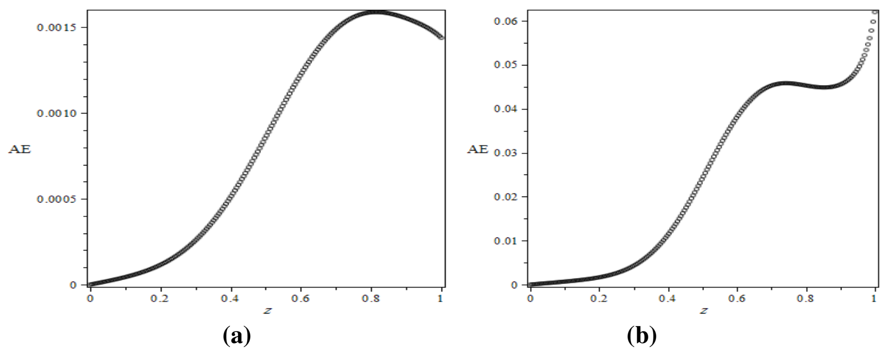


Fig. 4. Plot of the absolute errors of the CVIM solutions, (a) when $\alpha \approx 2.32$ and (b) when $\alpha \approx 6.11$

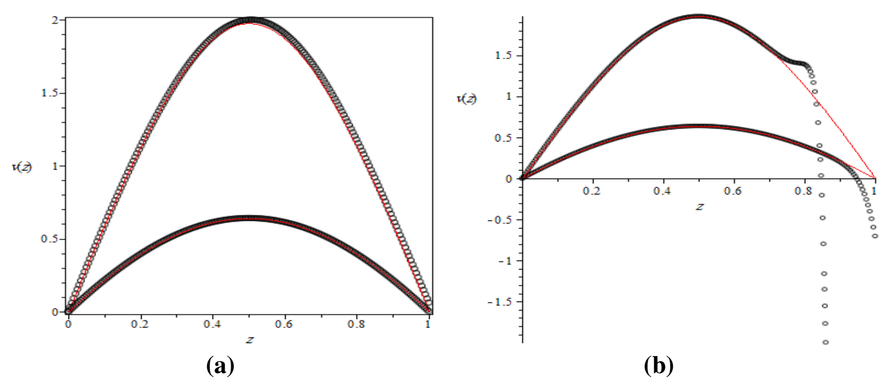


Fig. 5. A comparison between CVIM and VIM solutions (points) along with the exact solutions (lines), (a) CVIM results and (b) standard VIM results

5 Conclusion

In this paper, new formulation of Bratu equation that appears in electro-spinning process is presented. The proposed presentation eliminates the complexity that may appear when solving using the standard VIM. Moreover, the controlled VIM is proposed as well in order to increase the accuracy and control the convergence of the solution. The auxiliary parameter χ plays an important role in increasing the overall solution accuracy. From the obtained numerical results that illustrated graphically, one can conclude that the proposed CVIM display a high accuracy and reliability when compared with the standard VIM.

Competing Interests

Author has declared that no competing interests exist.

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