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Semigroups Connected with Parameter-elliptic Douglis-nirenberg Systems

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Abstract

It was shown by Seeley that associated with a parameter-elliptic boundary problem involving a system of differential operators of homogeneous type there was associated an analytic semigroup. This result was extended by Dreher to a Douglis-Nirenberg system of mono-order type, i.e., the diagonal operators are all of the same order. In this paper we again discuss the problem considered by Dreher, but use a different approach as his approach gives rise to certain difficulties. We also extend the results for mono-order systems to a certain class of Douglis-Nirenberg systems of multi-order type, i.e., the diagonal operators are not all of the same order. 2010 Mathematics Subject Classification: 35J55; 47D06

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1 Introduction

Several decades ago Seeley (1) considered a boundary problem for a $q \times q$ parameter-elliptic system of differential operators of homogeneous type defined over a compact manifold G with boundary. He denoted by A_B the operator on $L_p(G)^q$ induced by the boundary problem under null boundary conditions. Then with the implicit use of a priori estimates, he proceeded to show that, after a possible rotation and a shift in the spectral parameter, the fractional powers of A_B^{-1} formed an analytic semigroup.

Recently an extension of Seeley's work was undertaken by Dreher (2). He considered a boundary problem for a parameter-elliptic mono-order Douglis-Nirenberg system of differential operators over a bounded region $\Omega \subset \mathbb{R}^n$, where by mono-order we mean that the diagonal operators are all of the same order. Then, like Seeley, Dreher poses the problem in a suitable Sobolev space setting based on $L_p(\Omega)$ and denotes by $\mathcal{A}$ the operator acting in the Sobolev space just cited induced by the boundary problem under null boundary conditions. He then points out that in the mono-order case,

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the use of a priori estimates do not suffice in allowing one to establish that the negative powers of the operator $\mathcal{A}$ form an analytic semigroup; and hence in order to achieve this end he proceeds in an indirect manner. This involves in particular the use of his Lemma 3.1 wherein certain inequalities are presented which he claims follow from the results given in a certain reference. However an inspection of this and other references show that there are no such results, and hence since Lemma 3.1 is instrumental for the proof of the semigroup property of $\mathcal{A}$ (see his Theorem 2.4), one is led to question the validity of Drehers' results.

Accordingly, it is our opinion that the problems just cited which arise in Dreher's approach to the multi-order boundary problem are due to the fact that the problem is posed in a Lebesgue-Sobolev space setting. Thus, motivated by the paper (1), we feel that if one adequately chooses the space in which the boundary problem should be posed, then the a priori estimates for solutions should suffice in establishing the semigroup property. Hence the first objective of this paper is to present this new approach to the multi-order boundary problem. And this will be achieved by posing the problem in both a Bessel-potential and Sobolev space setting and then by appealing to a priori estimates for solutions of the boundary problem in this setting. These a priori estimate are proved in Theorem 3.1 and to our knowledge such kinds of estimates have not hitherto been established. The second objective of this paper will be to extend the foregoing works to a certain class of parameter-elliptic multi-order Douglis-Nirenberg systems defined over a bounded region $\Omega \subset \mathbb{R}^n$, i.e., where now the diagonal operators are not all of the same order. The class under investigation will be that class which was the subject of investigation in (3). And it is precisely the a priori estimates that were established there which will allow us to arrive, after a possible rotation and a shift in the spectral parameter, at the required semigroup property.

Accordingly, with these objectives in mind, let $N \in \mathbb{N}$ with $N > 1$ and let $\{ s_j \}_1^N, \{ t_j \}_1^N$, and $\{ \sigma_j \}_1^{N_0}$ be sequences of integers , where with $m_j = s_j + t_j$ for $j = 1, \ldots, N, N_0 = \frac{1}{2}\sum_{j=1}^N m_j$, and where in the sequel we will impose conditions which will ensure that $\sum_{j=1}^N m_j$ is even. Then we shall be concerned here with the boundary problem

$$A(x, D)\, u(x) - \lambda\, u(x) = f(x) \text{ in } \Omega, \tag{1.1}$$

$$B_j(x, D)\, u(x) = g_j(x) \text{ on } \Gamma \text{ for } j = 1, \ldots, N_0, \tag{1.2}$$

where Ω is a bounded region in $\mathbb{R}^n, n \geq 2$, with boundary Γ, $u(x) = (u_1(x), \ldots,$
$u_N(x))^T$, and $f(x) = (f_1(x), \ldots, f_N(x))^T$ are $N \times 1$ matrix functions defined on Ω, ^{T} denotes transpose, the $g_j(x)$ are scalar functions defined on Γ, $A(x, D)$ is an $N \times N$ matrix operator whose entries $A_{jk}(x, D)$ are linear operators defined on Ω of order not exceeding $s_j + t_k$ and defined to be 0 if $s_j + t_k < 0$, and $B_j(x, D), 1 \leq j \leq N_0$ is a $1 \times N$ matrix operator whose entries ar

e linear differential operators defined on Γ of order not exceeding $\sigma_j + t_k$ and defined to be zero if $\sigma_j + t_k < 0$. Our assumptions concerning the boundary problem (1.1), (1.2), which will depend upon each of the aformentioned boundary problems under consideration, will be made precise in the sequel.

2 Prelimimaries

In this section we are now going to introduce some terminology, definitions, and assumptions concerning the boundary problem (1.1), (1.2) which we require for our work.

Accordingly, we let $x = (x_1, \ldots, x_n) = (x', x_n)$ denote a generic point of $\mathbb{R}^n$ and use the notation $D_j = (-i\partial/\partial x_j, D = (D_1, \ldots, D_n), D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n} = D'^{\alpha'} D_n^{\alpha_n}$, and $\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}$ for $\xi = (\xi_1, \ldots, \xi_n) = (\xi', \xi_n) \in \mathbb{R}^n$, where $\alpha = (\alpha_1, \ldots, \alpha_n) = (\alpha', \alpha_n)$ is a multi-index whose length $\sum_{j=1}^n \alpha_j$ is denoted by $|\alpha|$. Differentiation with respect to another variable, say $y \in \mathbb{R}^n$, instead of x, will be indicated by replacing $D, D^\alpha, D'^{\alpha'}$, and $D_n^{\alpha_n}$ by $D_y, D_y^\alpha, D'^{\alpha'}_{y'}$, and $D_{y_n}^{\alpha_n}$, respectively. For $1 < p < \infty, s \in \mathbb{N}_0 = \mathbb{N} \cup \{ 0 \}$, and G an open set in $\mathbb{R}^\ell, \ell \in \mathbb{N}$, we let $W_p^s(G)$ the Sobolev space of order s related to $L_p(G)$ and denote the norm in this space by $\| \cdot \|_{s,p,G}$, where $\| u \|_{s,p,G} = \left(\sum_{|\alpha|\leq s} \int_G |D^\alpha u(x)|^p \, dx\right)^{1/p}$ for $u \in W_p^s(G)$. In addition we shall at times equip $W_p^s(G)$ with norms depending upon a parameter $\lambda \in \mathbb{C} \setminus \{0\}$, namely for $1 \leq j \leq N$, the norms

$$|||u|||^{(j)}_{s,p,G} = \| u\|_{s,p,G} + |\lambda|^{s/m_j} \| u\|_{0,p,G} \text{ for } u \in W_p^s(G).$$

We also let $\overset{o}{W_p^s}(G)$ denote the closure of $C_0^\infty(G)$ in $W_p^s(G)$.

In the sequel we shall also at times deal with the Bessel-potential space $H_p^s(G)$ for $s \in \mathbb{Z}$ and equipped with either its ordinary norm $\| \cdot \|_{s,p,G}$ or with norms $||| \cdot |||^{(j)}_{s,p,G}$, $j = 1, \dots, N$, depending upon the parameter λ. We recall from (4, Section 1) and (5, pp.177 and 310) that if $u \in H_p^s(G)$ and $G = \mathbb{R}^n$, then $\| u\|_{s,p,\mathbb{R}^n} = \| F^{-1}\langle\xi\rangle^s Fu\|_{0,p,\mathbb{R}^n}$ and $|||u|||^{(j)}_{s,p,\mathbb{R}^n} = \| F^{-1}\langle\xi,\lambda\rangle^s_j Fu\|_{0,p,\mathbb{R}^n}$, while if $G \neq \mathbb{R}^n$, then $\| u\|_{s,p,G} = \inf \| v\|_{s,p,\mathbb{R}^n}$ and $|||u|||^{(j)}_{s,p,G} = \inf |||v|||^{(j)}_{s,p,\mathbb{R}^n}$, where the infimum is taken over all $v \in H_p^s(\mathbb{R}^n)$ for which $u = v\big|_G$, F denotes the Fourier transformation on $\mathbb{R}^n$ $(x \to \xi)$, $\langle\xi\rangle = \left(|\xi|^2 + 1\right)^{1/2}$, and $\langle\xi,\lambda\rangle_j = \left(|\xi|^2 + |\lambda|^{2/m_j}\right)^{1/2}$. Note also from (5, p.316) that when $s \in \mathbb{N}_0$, G is bounded, and the boundary of G is sufficiently smooth (as will always be supposed in our work below), then $H_p^s(G)$ and $W_p^s(G)$ coincide (up to equivalent norms), and hence at times in the sequel we will write $H_p^s(G)$ in place of $W_p^s(G)$ for such values of s. Lastly we let $\mathbb{R}^n_\pm = \{ x \in \mathbb{R}^n \big| x_n \gtrless 0 \}$, $\mathbb{R}_\pm = \{ t \in \mathbb{R} \big| t \gtrless 0 \}$, and denote by I_ℓ the $\ell \times \ell$ unit matrix.

Assume for the moment that the boundary Γ of Ω (see (1.1), (1.2)) is of class C^k for some $k \in \mathbb{N}$, and let $s \in \mathbb{Z}$ satisfy $1 \leq s \leq k$. Then for $G = \Omega$ or $G = \mathbb{R}^n_+$, the vectors $u \in W_p^s(G)$ have boundary values $v = u\Big|_{\partial G}$ and we denote the space of these boundary values by $W_p^{s-1/p}(\partial G)$ and denote by $\| \cdot \|_{s-1/p,p,\partial G}$ the norm in this space, where $\| v\|_{s-1/p,p,\partial G} = \inf \| u\|_{s,p,G}$ for $v \in W_p^{s-1/p}(\partial G)$ and the infimum is taken over all those $u \in W_p^s(G)$ for which $u\Big|_{\partial G} = v$ (see also (6, Section 2), (7, p.20) for further definitions of $W_p^{s-1/p}(\partial G)$). In addition we shall use norms depending upon the parameter $\lambda \in \mathbb{C} \setminus \{0\}$, namely for $1 \leq j \leq N$,

$$|||v|||^{(j)}_{s-1/p,p,\partial G} = \| v\|_{s-1/p,p,\partial G} + |\lambda|^{(s-1/p)/m_j} \| v\|_{0,p,\partial G} \text{ for } v \in W_p^{s-1/p}(\partial G),$$

where $\| \cdot \|_{0,p,\partial G}$ denotes the norm in $L_p(\partial G)$.

In the sequel we shall make use of the following three results (here we suppose that $1 \leq j \leq N$ and that $|\lambda| \geq 1$. Firstly from (6, Proposition 2.2) we have:
(1) If $k, s \in \mathbb{N}$ with $1 \leq k < s$, then

$$|\lambda|^{(s-k)/m_j} |||u|||^{(j)}_{k,p,G} \leq C_1 |||u|||^{(j)}_{s,p,G} \tag{2.1}$$

for every $u \in W_p^s(G)$, where the constant C_1 does not depend upon u and λ.
(2) Suppose that the boundary of Ω is of class C^r for some $r \in \mathbb{N}$ and let $s \in \mathbb{N}$ satisfy $1 \leq s \leq r$. Then for $G = \Omega$ or $G = \mathbb{R}^n_+$ and for all $u \in W_p^s(G)$, we have

$$|||\gamma u|||^{(j)}_{s-1/p,p,\partial G} \leq C_2 |||u|||^{(j)}_{s,p,G}, \tag{2.2}$$

where γu denotes the trace of u on ∂G and the constant C_2 does not depend upon u and λ.

Secondly, the following result is an immediate consequence of the duality arguments given in (3, Proof of Proposition 4.1).
(3) Suppose that $u \in H_p^s(G), 0 > s \in \mathbb{Z}$, and $\phi \in C_0^{-s}(\mathbb{R}^n)$. Then

$$|||\phi u|||^{(j)}_{s,p,G} \leq C_3 |||u|||^{(j)}_{s,p,G}, \tag{2.3}$$

where the constant C_3 does not depend upon u and λ.

Finally turning to the boundary problem (1.1), (1.2), we henceforth write

$$A_{jk}(x,D) = \sum_{|\alpha|\le s_j+t_k} a_\alpha^{jk}(x)D^\alpha \text{ for } x\in\Omega \text{ and } 1\le j,k\le N,$$
$$B_{jk}(x,D) = \sum_{|\alpha|\le \sigma_j+t_k} b_\alpha^{jk}(x)D^\alpha \text{ for } x\in\Gamma \text{ and } k=1,\dots,N, j=1,\dots,N_0. \quad (2.4)$$

Also for $\xi\in\mathbb{R}^n$ we let

$$\overset{o}{A}(x,\xi) = \left(\overset{o}{A}_{jk}(x,\xi)\right)_{j,k=1}^{N} \text{ for } x\in\overline{\Omega},$$
$$\overset{o}{B}(x,\xi) = \left(\overset{o}{B}_{jk}(x,\xi)\right)_{\substack{j=1,\dots,N_0\\ k=1,\dots,N}} \text{ for } x\in\Gamma,$$

where $\overset{o}{A}_{jk}(x,\xi)$ (resp. $\overset{o}{B}_{jk}(x,\xi)$) consists of those terms in $A_{jk}(x,\xi)$ (resp. $B_{jk}(x,\xi)$) whose orders are exactly s_j+t_k (resp. σ_j+t_k). We denote by $\overset{o}{B}_j(x,\xi)$ the j-th row of $\overset{o}{B}(x,\xi)$.

3 The mono-order case

In this section we are going to deal with the first of our objectives stated in Section 1, i.e., the mono-order case wherein all the m_j are equal. We will denote by m the common value of the m_j, and as in (2), suppose that $m>0$ and max$\{\sigma_j\}_1^{N_0} < m$. We will also put $\sigma^\dagger = 1+$ max$\{\sigma_j\}_1^{N_0}$ and write $\langle\xi,\lambda\rangle$, $|||\cdot|||_{s,p,G}$, and $|||\cdot|||_{s-1/p,p,\partial G}$ in place of $\langle\xi,\lambda\rangle_j$, $|||\cdot|||_{s,p,G}^{(j)}$, and $|||\cdot|||_{s-1/p,p,\partial G}^{(j)}$, respectively, for $j=1,\dots,N$ since now all the m_j are equal. Furthermore, by interchanging the rows and columns of $A(x,D)$ and by adding and subtracting constants, there is no loss of generality in making the following assumption.

Assumption 3.1. We will henceforth suppose that

$$0<t_1\ge t_2\ge\cdots\ge t_N=0 \text{ and } m>s_1\le s_2\le\cdots\le s_N=m.$$

Then motivated by the fact that $t_N=0$ and by the requirement of parameter-ellipticity which will be defined below, we are also led to make the following assumption.

Assumption 3.2. It will henceforth be supposed that $\sigma^\dagger\ge 1$.

3.1 The case $\sigma^\dagger \le s_1$

In this subsection we restrict ourselves to the case $\sigma^\dagger\le s_1$. Then let us fix our attention again upon the boundary problem (1.1), (1.2).

Assumption 3.3. It will henceforth be supposed that : (1) Γ is of class $C^{\sigma^\dagger+t_1}$; (2) for each pair j,k, $a_\alpha^{jk}\in C^{|\sigma^\dagger-s_j|}$ for $|\alpha|\le s_j+t_k$ if $\sigma^\dagger-s_j\ne 0$, while if $\sigma^\dagger-s_j=0$, the $a_\alpha^{jk}\in L_\infty(\Omega)$ for $|\alpha|<s_j+t_k$ and $a_\alpha^{jk}\in C(\overline{\Omega})$ for $|\alpha|=s_j+t_k$; and (3) for each pair j,k, $b_\alpha^{jk}\in C^{\sigma^\dagger-\sigma_j}(\Gamma)$ for $|\alpha|\le\sigma_j+t_k$.

Definition 3.4. Let $\mathcal{L}$ be a closed sector in the complex plane with vertex at the origin.Then the boundary problem (1.1), (1.2) will be called parameter-elliptic in $\mathcal{L}$ if the following conditions are satisfied.

(1) det $\left(\mathring{A}(x,\xi)-\lambda I_N\right)\ne 0$ for $(x,\xi)\in\overline{\Omega}\times\mathbb{R}^n$ and $\lambda\in\mathcal{L}$ if $|\xi|+|\lambda|\ne 0$.

(2) Let $x^0 \in \Gamma$. Assume that the boundary problem (1.1), (1.2) is rewritten in a local coordinate system associated with x^0; it is obtained from the original one by means of an affine transformation after which $x^0 \to 0$ and $\nu \to e_n$, where ν denote the interior normal to Γ at x^0 and $(e_1, \ldots, e_n)$ denotes the standard basis in $\mathbb{R}^n$. Then for $\xi' \in \mathbb{R}^{n-1}$ and $\lambda \in \mathcal{L}$ the boundary problem on the half-line

$$\overset{o}{A}(0, \xi', D_n)v(t) - \lambda\, v(t) = 0 \text{ for } t = x_n > 0,$$
$$\overset{o}{B}_j(0, \xi', D_n)v(t) = 0 \text{ at } t = 0 \text{ for } j = 1, \ldots, N_0,$$
$$|v(t)| \to 0 \text{ to } t \to \infty,$$

has only the trivial solution for $|\xi'| + |\lambda| \neq 0$.

Remark 3.1. It follows from (8, Proposition 2.2) that when Condition (1) of Definition 3.4 is satisfied, then mN is even.

Theorem 3.1. *Suppose that the boundary problem* (1.1), (1.2) *is parameter- elliptic in $\mathcal{L}$. Then there exists a $\lambda_0 = \lambda_0(p) > 0$ such that for $\lambda \in \mathcal{L}$ with $|\lambda| \geq \lambda_0$, the boundary problem has a unique solution $u \in \prod_{j=1}^{N} W_p^{\sigma^\dagger + t_j}(\Omega)$ for any $f \in \prod_{j=1}^{N} H_p^{\sigma^\dagger - s_j}(\Omega)$ and $g \in \prod_{j=1}^{N_0} W_p^{\sigma^\dagger - \sigma_j - 1/p}(\Gamma)$, and the a priori estimate*

$$\sum_{j=1}^{N} |||u_j|||_{\sigma^\dagger + t_j, p, \Omega} \leq C \left(\sum_{j=1}^{N} |||f_j|||_{\sigma^\dagger - s_j, p, \Omega} + \sum_{j=1}^{N_0} |||g_j|||_{\sigma^\dagger - \sigma_j - 1/p, p, \Gamma} \right) \tag{3.1}$$

holds, where the constant C does not depend upon the f_j, g_j, and λ. Conversely, the estimate (3.1) *also holds if $u \in \prod_{j=1}^{N} W_p^{\sigma^\dagger + t_j}(\Omega)$, $\lambda \in \mathcal{L}$ with $|\lambda| \geq \lambda_0$ and f and the g_j are defined by* (1.1). (1.2).

Remark 3.2. It is also important to note that the estimate (3.1) is 2-sided, i.e., an estimate reverse to (3.1) holds. We will elaborate on this point in the proof of the theorem.

Proof of Theorem 3.1. Comparing the assertions of the theorem with analogous ones made in other papers dealing with parameter-elliptic boundary problems (e.g. (6), (8), (9), and (10)), we see that the problem considered here is non-standard in that for at least one j, we have $\sigma^\dagger - s_j < 0$. For this reason we will briefly outline its proof; and in this endeavour we will suppose henceforth that the hypothesis of the theorem holds and will proceed in several steps. Furthermore, by employing a standard extension proceedure we can henceforth suppose that the $a_\alpha^{jk}(x)$ and $b_\alpha^{jk}(x)$ of (2.4) are defined on all of $\mathbb{R}^n$, are compactly supported, and satisfy the same smoothness assumptions as asserted in Assumption 3.3 for $\overline{\Omega}$ and Γ, respectively.

Step 1. Let $x^0 \in \Omega$ and let us fix out attention upon the differential equation

$$\overset{o}{A}(x^0, D)u(x) - \lambda\, u(x) = f(x) \text{ for } x \in \mathbb{R}^n \text{ and } \lambda \in \mathcal{L} \setminus \{0\}. \tag{3.2}$$

Then observing that for $\xi \in \mathbb{R}^n$ and $\lambda \in \mathcal{L}$ with $|\lambda| \geq \lambda^0$ for a fixed $\lambda^0 > 0$, we may write

$$\left(\overset{o}{A}(x^0, \xi) - \lambda\, I_N\right) = \text{diag}\left(\langle \xi, \lambda \rangle^{s_1}, \ldots, \langle \xi, \lambda \rangle^{s_N}\right) \times$$
$$\left(\overset{o}{A}(x^0, \xi\langle \xi, \lambda \rangle^{-1}) - \lambda\, \langle \xi, \lambda \rangle^{-m} I_N\right) \text{diag}\left(\langle \xi, \lambda \rangle^{t_1}, \ldots, \langle \xi, \lambda \rangle^{t_N}\right),$$

it follows from Definition 3.4 that

$$\left|\text{det}\left(\overset{o}{A}(x^0, \xi) - \lambda\, I_N\right)\right| \geq C\, \langle \xi, \lambda \rangle^{2N_0},$$

where here and below C denotes a generic constant which may vary from inequality to inequality, but in all cases it does not upon x^0, ξ, λ, and the variables u, f, and the g_j which will appear below. Furthermore, if we put $\left(\mathring{A}(x^0, \xi) - \lambda\, I_N\right)^{-1} = (\tilde{a}_{jk}(\xi, \lambda))_{j,k=1}^{N}$, then it is clear that the $\tilde{a}_{jk}(\xi, \lambda)$ are rational functions of their arguments, while it is also not difficult to verify that for any multi-index α whose entries are either 0 or 1, $\left|\xi^{\alpha} D_{\xi}^{\alpha} \tilde{a}_{jk}(\xi,\lambda)\right| \leq C\langle \xi, \lambda\rangle^{-t_j - s_k}$ for all $\xi \in \mathbb{R}^n$ whose components are all non-zero. Hence if in (3.2) we denote by u_j and f_j, $1 \leq j \leq N$, the components of u and f , respectively, then arguments completely analogous to those used in the proofs of (3, Propositions 3.1-2) give the following two results:

(1) Suppose that $u \in \prod_{j=1}^{N} W_p^{\sigma^{\dagger}+t_j}(\mathbb{R}^n)$ and (3.2) holds. Then

$$f \in \prod_{j=1}^{N} H_p^{\sigma^{\dagger}-s_j}(\mathbb{R}^n) \text{ and } \sum_{j=1}^{N} |||f_j|||_{\sigma^{\dagger}-s_j,p,\mathbb{R}^n} \leq C \sum_{j=1}^{N} |||u_j|||_{\sigma^{\dagger}+t_j,p,\mathbb{R}^n}.$$

(2) If $f \in \prod_{j=1}^{N} H_p^{\sigma^{\dagger}-s_j}(\mathbb{R}^n)$, then there exists the constant $\lambda^{\dagger} \geq \lambda^0$ such that for $\lambda \in \mathcal{L}$ with $|\lambda| \geq \lambda^{\dagger}$, the differential equation (3.2) has a unique solution $u \in \prod_{j=1}^{N} W_p^{\sigma^{\dagger}+t_j}(\mathbb{R}^n)$ and $\sum_{j=1}^{N} |||u_j|||_{\sigma^{\dagger}+t_j,p,\mathbb{R}^n}$ $\leq C \sum_{j=1}^{N} |||f_j|||_{\sigma^{\dagger}-s_j,p,\mathbb{R}^n}$.

Step 2. For $x^0 \in \Omega$ and $\delta > 0$, let $B_{\delta}(x^0)$ denote the open ball in $\mathbb{R}^n$ with centre x^0 and radius δ. Also let supp denote support. Then we can appeal to (2.1) and argue in a manner similar to the way we did in the proofs of (3, Propositions 4.1-2) to obtain the following two results.

(1) For any $\epsilon > 0$ and $x^0 > 0$ there exists a δ, $0 < \delta <$ dist$\{x^0, \Gamma\}$, and a $\lambda^0 > 0$ such that for $\lambda \in \mathcal{L}$ with $|\lambda| \geq \lambda^0$

$$\sum_{j=1}^{N} ||| \sum_{k=1}^{N} \left(A_{jk}(x,D) - \overset{o}{A}_{jk}(x^0,D) \right) u_k |||_{\sigma^{\dagger}-s_j,p,\Omega} \leq \epsilon \sum_{j=1}^{N} |||u_j|||_{\sigma^{\dagger}+t_j,p,\Omega}$$

for every $u \in \prod_{j=1}^{N} W_p^{\sigma^{\dagger}+t_j}(\Omega)$ such that supp $u \subset B_{\delta}(x^0)$.

(2) For any $\epsilon > 0$ and $x^0 \in \Omega$ there exists a δ, $0 < \delta <$ dist$\{x^0, \Gamma\}$, and a $\lambda^0 > 0$ such that if supp $\left(a_{\alpha}^{jk}(x) - a_{\alpha}^{jk}(x^0)\right) \subset B_{\delta}(x^0)$ for $|\alpha| = s_j + t_k$, $j,k = 1,\ldots,N$, and $\lambda \in \mathcal{L}$ with $|\lambda| \geq \lambda^0$, then the estimate

$$\sum_{j=1}^{N} \sum_{k=1}^{N} ||| \left(\overset{o}{A}_{jk}(x,D) - \overset{o}{A}_{jk}(x^0,D) \right) u_k |||_{\sigma^{\dagger}-s_j,p,\mathbb{R}^n} \leq \epsilon \sum_{j=1}^{N} |||u_j|||_{\sigma^{\dagger}+t_j,p,\mathbb{R}^n}$$

holds for every $u \in \prod_{j=1}^{N} W_p^{\sigma^{\dagger}+t_j}(\mathbb{R}^n)$.

Step 3. Suppose next that $x^0 \in \Gamma$ and that the boundary problem (1.1), (1.2) has been rewritten in terms of the local coordinates at x^0 as explained in Definition 3.4. Let us now fix our attention upon the problem in the half-space

$$\overset{o}{A}(0,D)u(x) - \lambda\, u(x) = f(x) \text{ for } x \in \mathbb{R}_+^n \text{ and } \lambda \in \mathcal{L} \setminus \{0\}. \tag{3.3}$$

Then in light of what was said in Step 1 above we can now argue as in the proofs of Propositions 3.3-4 of (3) to deduce the following two results.

(1) Suppose that $u \in \prod_{j=1}^{N} W_p^{\sigma^{\dagger}+t_j}(\mathbb{R}_+^n)$ and that (3.3) holds. Then $f \in$ $\prod_{j=1}^{N} H_p^{\sigma^{\dagger}-s_j}(\mathbb{R}_+^n)$ and $\sum_{j=1}^{N} |||f_j|||_{\sigma^{\dagger}-s_j,p,\mathbb{R}_+^n} \leq C \sum_{j=1}^{N} |||u_j|||_{\sigma^{\dagger}+t_j,p,\mathbb{R}_+^n}$.

(2) Suppose that $f \in \prod_{j=1}^{N} H^{\sigma^{\dagger}-s_j}(\mathbb{R}_+^n)$. Then there exists a $\lambda^0 > 0$ such that for $\lambda \in \mathcal{L}$ with $|\lambda| \geq \lambda^0$ the differential equation (3.3) has a solution $u \in \prod_{j=1}^{N} W_p^{\sigma^{\dagger}+t_j}(\mathbb{R}_+^n)$ and $\sum_{j=1}^{N} |||u_j|||_{\sigma^{\dagger}+t_j,p,\mathbb{R}_+^n} \leq$ $C \sum_{j=1}^{N} |||f_j|||_{\sigma^{\dagger}-s_j,p,\mathbb{R}_+^n}$.

Step 4. Suppose again that $x^0 \in \Gamma$. Then in order to use some results from (3) pertaining to this case, we are now going to rewrite the boundary problem (1.1), (1.2) in terms of the local coordinates at x^0 as specified there. Accordingly let $\{U, \phi\}$ be a chart on Γ such that $x^0 \in U, \phi(x^0) = 0$, and

ϕ is a diffeomorphism of class $C^{\sigma^\dagger+t_1}$ mapping U onto an open set in $\mathbb{R}^n$ with $\phi(U\cap\Omega)\subset\mathbb{R}^n_+$ and $\phi(U\cap\Gamma)\subset\mathbb{R}^{n-1}$. Then by means of the mapping ϕ we can now pass to local coordinates at x^0 and in terms of these local coordinates each of the operators $A_{jk}(x,D)$ and $B_{jk}(x,D)$ can be written as $\tilde{A}_{jk}(y,D_y)=\sum_{|\alpha|\le s_j+t_k}\tilde{a}^{jk}_\alpha(y)D^\alpha_y$ and $\tilde{B}_{jk}(y,D_y)=\sum_{|\alpha|\le \sigma_j+t_k}\tilde{b}^{jk}_\alpha(y)D^\alpha_y$, repectively, for $y\in\phi(U)$. We denote by $\overset{o}{\tilde{A}}_{jk}\ (y,D_y)$ and $\overset{o}{\tilde{B}}_{jk}\ (y,D_y)$ the principal parts of $\tilde{A}_{jk}(y,D_y)$ and $\tilde{B}_{jk}(y,D_y)$, respectively. Then arguments completely analogous to those used in the proofs of (3, Propositions 4.1 and 4.3) give the following two results.

(1) For $j,k=1,\ldots,N$, let $\tilde{A}_{jk}(y,D_y)$ be extended to all of $\mathbb{R}^n$ by putting $\tilde{a}^{jk}_\alpha(y)=0$ for $y\in\mathbb{R}^n\setminus\phi(U)$ and $|\alpha|\le s_j+t_k$. Then for any $\epsilon>0$ there exists a $\delta>0$ and a $\lambda^0>0$ such that for $\lambda\in\mathcal{L}$ with $|\lambda|\ge\lambda^0$,

$$\sum_{j=1}^{N}|||\sum_{k=1}^{N}\left(\tilde{A}_{jk}(y,D_y)-\overset{o}{\tilde{A}}_{jk}\ (0,D_y)\right)u_k|||_{\sigma^\dagger-s_j,p,\mathbb{R}^n_+}\le\epsilon\sum_{j=1}^{N}|||u_j|||_{\sigma^\dagger+t_j,p,\mathbb{R}^n_+}$$

for every $u\in\prod_{j=1}^N W_p^{\sigma^\dagger+t_j}(\mathbb{R}^n_+))$ such that supp $u\subset B_\delta(0)\subset\overline{B_\delta(0)}\subset\phi(U)$.

(2) For $j,k=1,\ldots,N$, let $\overset{o}{\tilde{A}}_{jk}\ (y,D_y)$ be extended to all of $\mathbb{R}^n$ by putting $\tilde{a}^{jk}_\alpha(y)=\tilde{a}^{jk}_\alpha(0)$ for $y\in\mathbb{R}^n\setminus\phi(U)$ and $|\alpha|=s_j+t_k$. Then for any $\epsilon>0$ there exists a δ, $0<\delta<$ dist$\{0,\partial\,\phi(U)\}$, and a $\lambda^0>0$ such that if supp $\left(\tilde{a}^{jk}_\alpha(y)-\tilde{a}^{jk}_\alpha(0)\right)\subset B_\delta(0)$ for $|\alpha|=s_j+t_k, j.k=1,\ldots,N$ and $\lambda\in\mathcal{L}$ with $|\lambda|\ge\lambda^0$, then the estimate

$$\sum_{j=1}^{N}\sum_{k=1}^{N}|||\left(\overset{o}{\tilde{A}}_{jk}\ (y,D_y)-\overset{o}{\tilde{A}}_{jk}\ (0,D_y)\right)u_k|||_{\sigma^\dagger-s_j,p,\mathbb{R}^n_+}\le\epsilon\sum_{j=1}^{N}|||u_j|||_{\sigma^\dagger+t_j,p,\mathbb{R}^n_+}$$

holds for every $u\in\prod_{j=1}^N W_p^{\sigma^\dagger+t_j}(\mathbb{R}^n_+)$.

Step 5. Suppose still that $x^0\in\Gamma$ and that the boundary problem (1.1), (1.2) has been rewritten in terms of the local coordinates at x^0 (see Definition 3.4). Let us now turn to the problem in the half-space

$$\overset{o}{A}\ (0,D)u(x)-\lambda\,u(x)=0 \text{ for } x\in\mathbb{R}^n_+ \text{ and } \lambda\in\mathcal{L}\setminus\{0\}, \tag{3.4}$$

$$\overset{o}{B}_j\ (0,D)u(x)=g_j(x) \text{ at } x_n=0, j=1,\ldots,N_0. \tag{3.5}$$

Then the following two results are standard and follow from (2.2) and minor modifications of the arguments given in any of the following works: (8, Section 6), (3, Section 3), (9, Section 2), and (11, Section 3).

(1) Suppose that $u\in\prod_{j=1}^N W_p^{\sigma^\dagger+t_j}(\mathbb{R}^n_+)$ and that (3.5) holds. Then $g=(g_1,\ldots,g_{N_0})^T\in\prod_{j=1}^{N_0}W^{\sigma^\dagger-\sigma_j-1/p}(\mathbb{R}^n_+)$ and

$$\sum_{j=1}^{N_0}|||g_j|||_{\sigma^\dagger-\sigma_j-1/p,p,\mathbb{R}^{n-1}}\le C\sum_{j=1}^{N}|||u_j|||_{\sigma^\dagger+t_j,p,\mathbb{R}^n_+}.$$

(2) There exists a $\lambda^0>0$ such that for $\lambda\in\mathcal{L}$ with $|\lambda|\ge\lambda^0$, the boundary problem (3.4), (3.5) has a unique solution $u=(u_1,\ldots,u_N)^T\in\prod_{j=1}^N W_p^{\sigma^\dagger+t_j}(\mathbb{R}^n_+)$ for every $g=(g_1,\ldots,g_{N_0})^T\in\prod_{j=1}^{N_0}W_p^{\sigma^\dagger-\sigma_j-1/p}(\mathbb{R}^{n-1})$ and the a priori estimate

$$\sum_{j=1}^{N}|||u_j|||_{\sigma^\dagger+t_j,p,\mathbb{R}^n_+)}\le C\sum_{j=1}^{N_0}|||g_j|||_{\sigma^\dagger-\sigma_j-1/p,p,\mathbb{R}^{n-1}}$$

holds.

Step 6. If we refer to Step 4 for notation and appeal to (2.1), (2.2), the results of Step 5, and to arguments similar to those given in the references cited in Step 5, then it is not difficult to establish the following two results.
(1) for $j = 1, \ldots, N_0$, and $k = 1, \ldots, N$, let $\tilde{B}_{jk}(y, D)$ be extended to all of $\mathbb{R}^n$ by putting $\tilde{b}_\alpha^{jk}(y) = 0$ for $y \in \mathbb{R}^n \setminus \phi(U)$ and $|\alpha| \leq \sigma_j + t_k$. Then for any $\epsilon > 0$ there exists a $\delta > 0$ and a $\lambda^0 > 0$ such that for $\lambda \in \mathcal{L}$ with $|\lambda| \geq \lambda^0$

$$\sum_{j=1}^{N_0} ||| \sum_{k=1}^{N} \left(\tilde{B}_{jk}(y, D_y) - \overset{o}{\tilde{B}}_{jk}(0, D_y) \right) \gamma u_k |||_{\sigma^\dagger - \sigma_j - 1/p, p, \mathbb{R}^{n-1}} \leq \epsilon \sum_{j=1}^{N} |||u_j|||_{\sigma^\dagger + t_j, p, \mathbb{R}^n_+}$$

for every $u \in \prod_{j=1}^{N} W_p^{\sigma^\dagger + t_j}(\mathbb{R}^n_+)$ such that supp $u \subset B_\delta(0) \subset \overline{B_\delta(0)} \subset \phi(U)$, where γ denotes the trace operator.

(2) For $j = 1, \ldots, N_0$, $k = 1, \ldots, N$, let $\overset{o}{\tilde{B}}_{jk}(y, D_y)$ be extended to all of $\mathbb{R}^n$ by putting $\tilde{b}_\alpha^{jk}(y) = \tilde{b}_\alpha^{jk}(0)$ for $y \in \mathbb{R}^n \setminus \phi(U)$ and $|\alpha| = \sigma_j + t_k$. Then for any $\epsilon > 0$ there exists a δ, $0 < \delta < \text{dist}\{ 0, \partial\phi(U) \}$, and a $\lambda^0 > 0$ such that if supp $\left(\tilde{b}_\alpha^{jk}(y) - \tilde{b}_\alpha^{jk}(0) \right) \subset B_\delta(0)$ for $|\alpha| = \sigma_j + t_k, j = 1, \ldots, N_0, k = 1, \ldots, N$ and $\lambda \in \mathcal{L}$ with $|\lambda| \geq \lambda^0$, then the estimate

$$\sum_{j=1}^{N_0} \sum_{k=1}^{N} ||| \left(\overset{o}{\tilde{B}}_{jk}(y, D_y) - \overset{o}{\tilde{B}}_{jk}(0, D_y) \right) \gamma u_k |||_{\sigma^\dagger - \sigma_j - 1/p, p, \mathbb{R}^{n-1}} \leq \epsilon \sum_{j=1}^{N} |||u_j|||_{\sigma^\dagger + t_j, p, \mathbb{R}^n}$$

holds for every $u \in \prod_{j=1}^{N} W_p^{\sigma^\dagger + t_j}(\mathbb{R}^n)$.

Step 7. Finally, the proof of the theorem follows directly from (2.1)-(2.3), the results of Steps 1-6, and from the arguments use in the proofs of Lemma 4.1, Propositions 5.1 and 5.2, and Theorems 4.1 and 5.1 of (8). Furthermore, The assertions made in Remark 3.2 can be proved in exactly the same way. Indeed we know from (8) that we can cover $\overline{\Omega}$ by a finite number of open sets $\{ U_k \}_1^{n_1}$, where $U_k \cap \Gamma \neq \emptyset$ for $k \leq n_0 < n_1$, and $U_k \subset \Omega$ for $k > n_0$. If $\{ \phi_k \}_1^{n_1}$ denotes a partition of unity subordinate to the covering $\{ U_k \}_1^{n_1}$ such that supp $\phi_k \cap \Gamma \neq \emptyset$ for $k \leq n_0$ and supp $\phi_k \cap \Gamma = \emptyset$ otherwise, then a norm equivalent to the norm $|||f_j|||_{\sigma^\dagger - s_j, p, \Omega}$ defined above is given by $\sum_{k=1}^{n_0} |||\phi_j f_j|||_{\sigma^\dagger - s_j, p, \mathbb{R}^n_+} + \sum_{k=n_0+1}^{n_1} |||\phi_k f_j|||_{\sigma^\dagger - s - j, p, \mathbb{R}^n}$, where it is to be understood that in the first summation the norms $|||\phi_k f_j|||_{\sigma^\dagger - s_j, p, \mathbb{R}^n_+}$ are taken in local coordinates. Since similar statements can be made for both $|||u_j|||_{\sigma^\dagger + t_j, p, \Omega}$ and $|||g_j|||_{\sigma^\dagger - \sigma_j - 1/p, p, \Gamma}$, the assertion of Remark 3.2 follows directly from (2.1)-(2.3) and the results of Steps 1-6. □

Notation. (1) We have so far equipped the spaces $W_p^s(\Omega)$ for $s \in \mathbb{N}_0$ and $H_p^s(\Omega)$ for $s \in \mathbb{Z}$ with either their ordinary norms or norms depending upon the parameter λ. To distinguish between these two cases let us henceforth denote these spaces by $W_{p,\lambda}^s(\Omega)$ and $H_{p,\lambda}^s(\Omega)$, respectively, when they are equipped with their λ dependent norms, so that the notation $W_p^s(\Omega)$ and $H_p^s(\Omega)$ will from now on mean that these two spaces are equipped with their ordinary norms. We also let

$$W_p^{(\sigma^\dagger+t,\lambda)}(\Omega) = \prod_{j=1}^N W_{p,\lambda}^{\sigma^\dagger+t_j}(\Omega), \qquad W_p^{(\sigma^\dagger+t)}(\Omega) = \prod_{j=1}^N W_p^{\sigma^\dagger+t_j}(\Omega),$$

$$H_p^{(\sigma^\dagger-s,\lambda)}(\Omega) = \prod_{j=1}^N H_{p,\lambda}^{\sigma^\dagger-s_j}(\omega), \qquad H_p^{(\sigma^\dagger-s)}(\Omega) = \prod_{j=1}^N H_p^{\sigma^\dagger-s_j}(\Omega).$$

(2) In the sequel we will require the following notation. If $X = \prod_{j=1}^r X_j$, where the X_j denote Banach spaces equipped with norms $\|\cdot\|_{X_j}$, and $f = (f_1, \ldots, f_r) \in X$, then we let $\| f \|_X = \sum_{j=1}^r \| f_j \|_{X_j}$ (note that at times in the sequel we will replace the norms $\|\cdot\|_{X_j}$ and $\|\cdot\|_X$ by the norms $|||\cdot|||_{X_j}$ and $|||\cdot|||_X$, respectively, to indicate that we are now considering the parameter dependent norms introduced above). In addition, if X and Y denote Banach spaces and T a continuous linear operator from X into Y, then we will use the notation $\| T \|_{X\to Y}$ (or $|||T|||_{X\to Y}$) to denote the norm of this operator.

For the remainder of this subsection we will suppose that the hypothesis of Theorem 3.1 holds and denote by A_p the operator on $H_p^{(\sigma^\dagger-s)}(\Omega)$ that acts as $A(x, D)$ and has domain

$$D(A_p) = \Big\{ u \in W_p^{(\sigma^\dagger+t)}(\Omega) \Big| B_j(x, D)u(x) = 0 \text{ for } x \in \Gamma, j = 1, \ldots, N_0 \Big\}. \tag{3.6}$$

As a consequence of Theorem 3.1 it follows that if $\lambda \in \mathcal{L}$ with $|\lambda| \geq \lambda_0$ and if $D(A_p)$ is equipped with the norm $|||\cdot|||_{W_p^{(\sigma^\dagger+t,\lambda)}(\Omega)}$, then the mapping $(A_p - \lambda I_N) : D(A_p) \to H_p^{(\sigma^\dagger-s,\lambda)}(\Omega)$ is an isomorphism and $|||A_p - \lambda I_N|||_{D(A_p)\to H_p^{(\sigma^\dagger-s,\lambda)}(\Omega)} \leq C$. Thus we see that as an operator in $H_p^{(\sigma^\dagger-s)}(\Omega), A_p$ has a non-empty resolvent set. Furthermore, if we let $R_p(\lambda)$ denote its resolvent, then it follows that for $f \in H_p^{(\sigma^\dagger-s)}(\Omega)$ and $\lambda \in \mathcal{L}$ with $|\lambda| \geq \lambda_0$, we have

$$|||R_p(\lambda)|||_{W_p^{(\sigma^\dagger+t,\lambda)}(\Omega)} \leq C|||f|||_{H_p^{(\sigma^\dagger-s,\lambda)}(\Omega)} \leq C\| f \|_{H_p^{(\sigma^\dagger-s)}(\Omega)}. \tag{3.7}$$

Proposition 3.1. *Suppose that* $\lambda \in \mathcal{L}$ *with* $|\lambda| \geq \lambda_0$. *Then as an operator from* $H_p^{(\sigma^\dagger-s)}(\Omega)$ *into itself,* $R_p(\lambda)$ *is compact and* $|\lambda|\| R_p(\lambda) \|_{H_p^{(\sigma^\dagger-s)}(\Omega)\to H_p^{(\sigma^\dagger-s)}(\Omega)} \leq C$.

Proof. Let X and Y be Banach spaces such that $X \subset Y$, with continuous embedding, and let $I_{X\to Y}$ denote the corresponding embedding operator. Then for $1 \leq j \leq N$, we have

$$I_{W_{p,\lambda}^{\sigma^\dagger+t_j}(\Omega)\to H_p^{\sigma^\dagger-s_j}(\Omega)} = I_{L_p(\Omega)\to H_p^{\sigma^\dagger-s_j}(\Omega)} I_{W_{p,\lambda}^{\sigma^\dagger+t_j}(\Omega)\to L_p(\Omega)};$$

and it is clear that $I_{W_{p,\lambda}^{\sigma^\dagger+t_j}(\Omega)\to L_p(\Omega)}$ is bounded in norm by $C|\lambda|^{-(\sigma^\dagger+t_j)/m}$, while a duality argument also shows that $I_{L_p(\Omega)\to H_p^{\sigma^\dagger-s_j}(\Omega)}$ is bounded in norm by $C|\lambda|^{(\sigma^\dagger-s_j)/m}$. Since the mapping $I_{W_p^{\sigma^\dagger+t_j}(\Omega)\to L_p(\Omega)}$ is compact and since we can write $(R_p(\lambda)f)_j = I_{W_{p,\lambda}^{\sigma^\dagger+t_j}(\Omega)\to L_p(\Omega)} (R_p(\lambda)f)_j$ for $f \in H_p^{(\sigma^\dagger-s)}(\Omega)$, where $(R_p(\lambda)f)_j$ denotes the j-th component of $R_p(\lambda)f$, all the assertions of the proposition follow from these results and (3.7). □

It follows from the foregoing results that A_p is a closed, densely defined operator on $H_p^{(\sigma^\dagger-s)}(\Omega)$. Furthermore, we have just seen that A_p has a compact resolvent, and hence a discrete spectrum, so that at most a finite number of eigenvalues of A_p lie in $\mathcal{L}$. Thus by a shift in the spectral parameter if necessary, we see that there is no loss of generality in supposing henceforth that 0 lies in the resolvent set of A_p. Next for $0 < \theta \leq \pi$ let $\mathcal{L}_\theta$ denote the closed sector in the λ-plane with vertex at the origin defined by the inequalities $\theta \leq |\arg\lambda| \leq \pi$. Then it is clear that all the assumptions and results presented so far for the boundary problem (1.1), (1.2) remain perfectly valid for the boundary problem (1.1)$'$,(1.2), where (1.1)$'$ is obtained from (1.1) by replacing $A(x, D)$ by $\rho\, A(x, D)$ and λ by $\rho\, \lambda$, where ρ denotes a constant satisfying $|\rho| = 1$. Thus we see that by an appropriate choice of θ, we are led to the following asumption.

Assumption 3.5. We henceforth supose that the sector $\mathcal{L}$ of Theorem 3.1 coincides with the sector $\mathcal{L}_\theta$ introduced above and that $\mathcal{L}_\theta \cup \tilde{B}_{2\epsilon}(0)$ is contained in the resolvent set of A_p for some $\epsilon > 0$, where $\tilde{B}_{2\epsilon} = \{\, \lambda \in \mathbb{C} \,\big|\, |\lambda| < 2\epsilon \,\}$.

Next let γ denote the contour in the λ-plane consisting of the segment $r\, e^{i\theta}$ with r running from ∞ to ϵ, the arc $\epsilon\, e^{i\phi}$ with ϕ running from θ to $-\theta$, and the segment $r\, e^{i\theta}$ with r running from ϵ to ∞. Then we come to the main result of this subsection.

Theorem 3.2. *Suppose that the hypothesis of Theorem 3.1 as well as Assumption 3.5 hold. Then the boundary problem* (1.1)*,* (1.2) *generates a semigroup of operators on* $H_p^{(\sigma^\dagger - s)}(\Omega)$*, namely* $A_p^{-t} = -(2\pi\, i)^{-1} \int_\gamma \lambda^{-t} R_p(\lambda)\, d\lambda,\ t \geq 0$*. Furthermore,* A_p^{-t} *is analytic in the half-space Re* $t > 0$*.*

Proof. Bearing in mind Proposition 3.1 and the fact that the contour γ can be suitably deformed (see (6, Subsection 6.2)), the assertions of the theorem follow directly from the results given in (12, Subsection 14.2, p.280). □

3.2 The case $\sigma^\dagger > s_1$

In this subsection we restrict ourselves to the case $\sigma^\dagger > s_1$. Then for this case we have $\sigma^\dagger - s_1 > 0$ and $\sigma^\dagger - s_N \leq 0$, and we henceforth let k_1 = max $\{\, k \,\big|\, 1 \leq k < N,\ \sigma^\dagger - s_j > 0 \,\}$. It would be tempting to treat this case in the same way as we treated the case $\sigma^\dagger \leq s_1$. Indeed, if for this case we also suppose as in Theorem 3.1 that the boundary problem (1.1).(1.2) is parameter-elliptic in $\mathcal{L}$, then we could argue as we did in the proof of Theorem 3.1 to show that all the assertions of that theorem remain valid. Furthermore, if we now define A_p in an analogous fashion to that in Subsection 3.1 and employ the terminology introduced there, then we can also show that if $\lambda \in \mathcal{L}$ with $|\lambda| \geq \lambda_0$ and $D(A_p)$ is equipped with the norm $|||\cdot|||_{W_p^{(\sigma^\dagger + t, \lambda)}(\Omega)}$, then the mapping $A_p - \lambda\, I_N : D(A_p) \to H_p^{(\sigma^\dagger - s, \lambda)}(\Omega)$ is an isomorphism and $R_p(\lambda)$ satisfies the first inequality of (3.7), but not the second, since $\sigma^\dagger - s_j > 0$ for $1 \leq j \leq k_1$. Thus unlike the case $\sigma^\dagger \leq s_1$, we can no longer take $H_p^{(\sigma^\dagger - s)}(\Omega)$ as the basic space (i.e., a space that is equipped with a norm not depending upon λ) on which to construct an analytic semigroup of operators. To overcome this problem we are led to consider the boundary problem

$$\tilde{A}(x, D)u(x) - \lambda\, u(x) = f(x) \text{ in } \Omega, \tag{3.8}$$

$$\tilde{B}_j(x, D) = g_j \text{ on } \Gamma \text{ for } j = 1, \ldots, (N_0 - \tilde{N}_0), \tag{3.9}$$

where $\tilde{A}(x, D) = (A_{k_1+j, k_1+k}(x, D))_{j,k=1}^{N-k_1}$, $\tilde{B}_j(x, D) = \big(B_{\tilde{N}_0+j, k_1+1}, \ldots, B_{\tilde{N}_0+j, N}\big)$, and $\tilde{N}_0 = mk_1/2$ (we will impose conditions below which will ensure that mk_1 is even).

Assumption 3.6. It will henceforth be supposed that: (1) Γ is of class $C^{\sigma^\dagger + t_{k_1+1}}$; (2) Condition (2) of Assumption 3.3 holds for $k_1 + 1 \leq j, k \leq N$, while for other values of $j, k,\ a_{jk}^\alpha \in L_\infty(\Omega)$ and $a_{jk}^\alpha \in C(\overline{\Omega})$ for $|\alpha| = s_j + t_k,\ 1 \leq j, k \leq k_1$; and (3) Condition (3) of Assumption 3.3 holds for $\tilde{N}_0 < j \leq N_0$ and $k_1 < k \leq N$, while for other values of $j, k,\ b_\alpha^{jk} \in L_\infty(\Gamma)$.

For $\xi \in \mathbb{R}^n$ let us now put

$$\overset{o}{A}{}^{\dagger}(x,\xi) = \left(\overset{o}{A}_{jk}(x,\xi)\right)_{j,k=1}^{k_1} \text{ for } x \in \overline{\Omega},$$
$$\overset{o}{\tilde{A}}(x,\xi) = \left(\overset{o}{A}_{k_1+j,k_1+k}(x,\xi)\right)_{j,k=1}^{N-k_1} \text{ for } x \in \overline{\Omega},$$
$$\overset{o}{\tilde{B}}_j(x,\xi) = \left(\overset{o}{B}_{\tilde{N}_0+j,k_1+1}(x,\xi),\ldots,\overset{o}{B}_{\tilde{N}_0+j,N}(x,\xi)\right) \text{ for } x \in \Gamma \text{ and } j = 1,\ldots,(N_0-\tilde{N}_0).$$

Definition 3.7. Let $\mathcal{L}$ be a closed sector in the complex plane with vertex at the origin. Then the boundary problem (3.8), (3.9) will be called parameter-elliptic in $\mathcal{L}$ if the following conditions are satisfied.

(1) $\det\left(\overset{o}{A}{}^{\dagger}(x,\xi) - \lambda I_{k_1}\right) \neq 0$ and $\det\left(\overset{o}{\tilde{A}}(x,\xi) - \lambda I_{N-k_1}\right) \neq 0$ for $(x,\xi) \in \overline{\Omega} \times \mathbb{R}^n$ and $\lambda \in \mathcal{L}$ if $|\xi| + |\lambda| \neq 0$.

(2) Let $x^0 \in \Gamma$ and assume that the boundary problem (3.8), (3.9) is rewritten in a local coordinate system associated with x^0 as explained in Definition 3.4. Then for $\xi' \in \mathbb{R}^{n-1}$ and $\lambda \in \mathcal{L}$ the boundary problem on the half-line

$$\overset{o}{\tilde{A}}(0,\xi',D_n)v(t) - \lambda v(t) = 0 \text{ for } t = x_n > 0,$$
$$\overset{o}{\tilde{B}}_j(0,\xi',D_n)v(t) = 0 \text{ at } t = 0 \text{ for } j = 1,\ldots,N_0-\tilde{N}_0,$$
$$|v(t)| \to 0 \text{ as } t \to \infty,$$

has only the trivial solution for $|\xi'| + |\lambda| \neq 0$.

Remark 3.3. Bearing in mind Remark 3.1, it follows from the arguments of (8) that if Condition (1) of Definition 3.7 is satisfied, then mk_1 is even.

Arguments analogous to those used in the proof of Theorem 3.1 give the following result.

Theorem 3.3. *Suppose that the boundary problem* (3.8), (3.9) *is parameter-elliptic in* $\mathcal{L}$. *Then there exists a* $\lambda_0 = \lambda_0(p) > 0$ *such that for* $\lambda \in \mathcal{L}$ *with* $|\lambda| \geq \lambda_0$, *the boundary problem has a unique solution* $u \in \prod_{j=1}^{N-k_1} W_p^{\sigma^\dagger + t_{k_1+j}}(\Omega)$ *for any* $f \in \prod_{j=1}^{N-k_1} H_p^{\sigma^\dagger - s_{k_1+j}}(\Omega)$ *and* $g \in \prod_{j=1}^{N_0-\tilde{N}_0} W_p^{\sigma^\dagger - \sigma_{\tilde{N}_0+j} - 1/p}(\Gamma)$, *and the a priori estimate*

$$\sum_{j=1}^{N-k_1} |||u_j|||_{\sigma^\dagger + t_{k_1+j},p,\Omega} \leq C\left(\sum_{j=1}^{N-k_1} |||f_j|||_{\sigma^\dagger - s_{k_1+j},p,\Omega} + \sum_{j=1}^{N_0-\tilde{N}_0} |||g_j|||_{\sigma^\dagger - \sigma_{\tilde{N}_0+j} - 1/p,p,\Gamma}\right)$$

holds.

Note that in the above theorem C denotes the generic constant introduced in the text following (3.2).

Next, bearing in mind the notation following the proof of Theorem 3.1, let us now put

$$\tilde{W}_p^{(\sigma^\dagger+t,\lambda)}(\Omega) = \prod_{j=1}^{N-k_1} W_{p,\lambda}^{\sigma^\dagger + t_{k_1+j}}(\Omega), \qquad \tilde{W}_p^{(\sigma^\dagger+t)}(\Omega) = \prod_{j=1}^{N-k_1} W_p^{\sigma^\dagger + t_{k_1+j}}(\Omega),$$
$$\tilde{H}_p^{(\sigma^\dagger-s,\lambda)}(\Omega) = \prod_{j=1}^{N-k_1} H_{p,\lambda}^{\sigma^\dagger - s_{k_1+j}}(\Omega), \qquad \tilde{H}_p^{(\sigma^\dagger-s)}(\Omega) = \prod_{j=1}^{N-k_1} H_p^{\sigma^\dagger - s_{k_1+j}}(\Omega).$$

Also for the remainder of this subsection we will suppose that the hypothesis of Theorem 3.3 holds and denote by $\tilde{A}_p$ the operator in $\tilde{H}_p^{(\sigma^\dagger - s)}(\Omega)$ which acts as $\tilde{A}(x, D)$ and has domain

$$D(\tilde{A}_p) = \left\{ u \in \tilde{W}_p^{(\sigma^\dagger + t)}(\Omega) \,\middle|\, \tilde{B}_j(x, D)u(x) = 0 \text{ on } \Gamma \text{ for } j = 1, \ldots, N_0 - \tilde{N}_0 \right\}.$$

Then as a consequence of Theorem 3.3, it follows that if $\lambda \in \mathcal{L}$ with $|\lambda| \geq \lambda_0$ and $D(\tilde{A}_p)$ is equipped with the norm $|||\cdot|||_{\tilde{W}_p^{(\sigma^\dagger + t, \lambda)}(\Omega)}$, then the mapping $\left(\tilde{A}_p - \lambda\, I_{N-k_1}\right) : D(\tilde{A}_p) \to \tilde{H}_p^{(\sigma^\dagger - s, \lambda)}(\Omega)$ is an isomorphism and
$|||\tilde{A}_p - \lambda\, I_{N-k_1}|||_{D(\tilde{A}_p) \to \tilde{H}^{(\sigma^\dagger - s, \lambda)}(\Omega)} \leq C$. Thus we see that as an operator on $\tilde{H}_p^{(\sigma^\dagger - s)}(\Omega), \tilde{A}_p$ has a non-empty resolvent set. Furthermore, if we let $\tilde{R}_p(\lambda)$ denote its resolvent, then it follows that for $f \in \tilde{H}_p^{(\sigma^\dagger - s)}(\Omega)$ and $\lambda \in \mathcal{L}$ with $|\lambda| \geq \lambda_0$, we have

$$|||\tilde{R}_p(\lambda) f|||_{\tilde{W}_p^{(\sigma^\dagger + t, \lambda)}(\Omega)} \leq C |||f|||_{\tilde{H}_p^{(\sigma^\dagger - t, \lambda)}(\Omega)} \leq C \| f \|_{\tilde{H}_p^{\sigma^\dagger - s)}(\Omega)}.$$

In addition we can argue as in the proof of Proposition 3.1 to show that for $\lambda \in \mathcal{L}$ with $|\lambda| \geq \lambda_0$, the operator $\tilde{R}_p(\lambda) : \tilde{H}_p^{(\sigma^\dagger - s)}(\Omega) \to \tilde{H}_p^{\sigma^\dagger - s)}(\Omega)$ is compact and $|\lambda| \|\tilde{R}_p(\lambda)\|_{\tilde{H}_p^{(\sigma^\dagger - s)}(\Omega) \to \tilde{H}_p^{\sigma^\dagger - s)}(\Omega)} \leq C$.

As a consequence of the foregoing results it follows that $\tilde{A}_p$ is a closed, densely defined operator in $\tilde{H}_p^{(\sigma^\dagger - s)}(\Omega)$ with compact resolvent, and hence a discrete spectrum. Then for reasons made in the text preceding Assumption 3.5, we are led to make the following assumption.

Assumption 3.8. We henceforth suppose that the sector $\mathcal{L}$ of Theorem 3.3 coincided with the sector $\mathcal{L}_\theta$ defined in the text preceding Assumption 3.5 and that $\mathcal{L}_\theta \cup \tilde{B}_{2\epsilon}(0)$ is contained in the resolvent set of $\tilde{A}_p$.

Finally, with γ denoting the contour defined in the text preceding Theorem 3.2, we can argue as in the proof of that theorem to obtain the following result.

Theorem 3.4. *Suppose that the hypothesis of Theorem 3.3 as well as Assumption 3.8 hold. Then the boundary problem* (3.8)*,* (3.9)*, which is associated with the boundary problem* (1.1)*,* (1.2)*, generates a semigroup of operators on* $\tilde{H}_p^{(\sigma^\dagger - s)}(\Omega)$*, namely* $\tilde{A}_p^{-t} = -(2\pi\, i)^{-1} \int_\gamma \lambda^{-t} \tilde{R}_p(\lambda)\, d\,\lambda,\ t \geq 0$. *Furthermore,* $\tilde{A}_p^{-t}$ *is analytic in the half-plane Re* $t > 0$.

3.3 An example

Let us now consider a boundary problem arising in quantum hydrodynamics that was discussed in (2) and which is covered by the theory expounded in Subsection 3.2. Here we have the boundary problem (1.1), (1.2) with Ω a bounded region in $\mathbb{R}^2$ with smooth boundary $\Gamma, N = 3$,

$$A(x, D) = \begin{pmatrix} -\nu_0 \Delta & -iD_1 & -iD_2 \\ i\frac{\epsilon^2}{4} D_1 \Delta & -\nu_0 \Delta & 0 \\ i\frac{\epsilon^2}{4} D_2 \Delta & 0 & -\nu_0 \Delta \end{pmatrix},$$

$B_1(x, D) = \partial/\partial\,\nu, B_2(x, D) = 1$, and $B_3(x, D) = 1$ for $x \in \Gamma$, where Δ denotes the Laplacian in $\mathbb{R}^2$, ν_0 and ϵ denote positive constants, and $\partial/\partial\,\nu$ denote differentiation along the interior normal to Γ. For this problem we take $t_1 = 1, t_2 = t_3 = 0, s_1 = 1, s_2 = s_3 = 2,\ \sigma^\dagger = 2$, and thus $s_j + t_j = m = 2$ for $j = 1, 2, 3$ and $\sigma^\dagger - s_1 = 1$.

Let us now take $\mathcal{L} = \mathcal{L}_\theta$ for $0 < \theta \leq \pi$. Then direct calculations show that Conditions (1) and (2) of definition 3.7 hold. Thus if we let $\tilde{A}_p$ denote the operator in $L_p(\Omega)^2$ that acts as

$$\begin{pmatrix} -\nu_0 \Delta & 0 \\ 0 & -\nu_0 \Delta \end{pmatrix}$$

and has domain $\left(W_p^2(\Omega)\cap \overset{o}{W}{}_p^{1}(\Omega)\right)^2$, then we know from the results given in Subsection 3.2 that $\tilde{A}_p$ is a densely defined, closed operator in $L_p(\Omega)^2$ with compact resolvent $\tilde{R}_p(\lambda)$, and hence a discrete spectrum.

We now assert that for some $\epsilon > 0, \mathcal{L}_\theta \cup B_\epsilon(0)$ is contained in the resolvent set of $\tilde{A}_p$. To see this let us fix our attention upon the boundary problem

$$-\nu_0 \Delta\, u(x) - \lambda\, u(x) = 0 \text{ for } x \in \Omega, \tag{3.10}$$

$$u(x) = 0 \text{ for } x \in \Gamma. \tag{3.11}$$

It is easy to verify that this boundary problem falls into the class of parameter-elliptic boundary studied in (6), and so it follows from (6, Theorem 2.1) that if we let $\mathcal{A}_p$ denote the operator in $L_p(\Omega)$ that acts as $-\nu_0\Delta$ and has domain $W_p^2(\Omega)\cap \overset{o}{W}{}_p^{1}(\Omega)$, then $\mathcal{A}_p$ is a closed, densely defined opertor with compact resolvent, and hence has a discrete spectrum, and that for some $\lambda^0 = \lambda^0(p) > 0$, the set $\{\, \lambda \in \mathcal{L}_\theta \,\big|\, |\lambda| \geq \lambda^0 \,\}$ is contained in the resolvent set of $\mathcal{A}_p$.

Now let us fix our attention upon the case $p = 2$. Then associated with the boundary problem (3.10), (3.11) is the sesquilinear form $B(u, v)$ on $L_2(\Omega)$ with domain $D(B) = \overset{o}{W}{}_2^{1}(\Omega)$, where $B(u, v) = \nu_0 \sum_{j=1}^{2}(D_j u, D_j v)$, and $(\cdot,\cdot)$ denotes the inner product in $L_2(\Omega)$. It is a simple matter to verify that B is a densely defined, symmetric, closed form which is bounded from below by 0 (see (13, p.310 and Theorem 2.1, p.322)). Furthermore we must have $B(u, u) \geq \delta > 0$ for $u \in D(B)$ with $\| u\|_{0,2,\Omega} = 1$, since otherwise there would exist a $u \in D(B)$, $\| u\|_{0,2,\Omega} = 1$ such that $B(u, u) = 0$, which leads to the contradiction that $u = 0$. Hence if we let $\tilde{\mathcal{A}}_2$ denote the selfadjoint operator associated with the form B (see (13, Theorem 2.6, p.223)), then $\tilde{\mathcal{A}}_2 \geq \delta$, which implies that the spectrum of $\tilde{\mathcal{A}}_2$ is contained in $[\delta, \infty)$. On the other hand we can argue as in (14, pp.107-112) to show that $\tilde{\mathcal{A}}_2 = \mathcal{A}_2$; and since we know from (15) and (10) that the spectrum of $\mathcal{A}_p$ does not depend upon p, it follows that the spectrum of $\mathcal{A}_p$ is contained in $[\delta, \infty)$. As a consequence of this last result it follows that the assertion that $\mathcal{L}_\theta \cup \tilde{B}_{2\epsilon}(0)$ lies in the reolvent set of $\tilde{A}_p$ is certainly true if we take $\epsilon < \delta/2$.

In light of the foregoing results and Theorem 3.4, we conclude that the operators $\tilde{A}_p^{-t}, t \geq 0$, form a semigroup in $L_p(\Omega)^2$ and that $\tilde{A}_p^{-t}$ is analytic in the half-space $t > 0$.

4 The multi-order case

In this section we fix our attention again upon the boundary problem (1.1), (1.2), but we now suppose that $s_1 \geq s_2 \geq \ldots \geq s_N$, $t_1 \geq t_2 \geq \ldots \geq t_N \geq 0$, and let $m_j = s_j + t_j$ for $j = 1, \ldots, N$. We also suppose that $m_1 = m_2 = \ldots = m_{k_1} > m_{k_1+1} = \ldots = \ldots m_{k_{d-1}} > m_{k_{d-1}+1} = \ldots = m_{k_d} > 0$, where $k_d = N$, put $\tilde{m}_j = m_{k_j}$ for $j = 1, \ldots d$, and let $\tilde{I}_r$ denote the $(k_r - k_{r-1}) \times (k_r - k_{r-1})$ identity matrix for $r = 1, \ldots, d$, where $k_0 = 0$. In the sequel we will impose conditions which will ensure that for $r = 1, \ldots, d$, the sum $\sum_{j=1}^{k_r} m_j$ is even and henceforth denote this sum by $2N_r$. Lastly, we suppose that max$\{\, \sigma_j \,\}_1^{N_0} < s_N$. Then as indicated in (3), there is no loss of generality in making the following assumption.

Assumption 4.1. It will henceforth be supposed that $t_j \geq 0$ and $s_j \geq 0$ for $j = 1, \ldots N$, and that $\sigma_j < 0$ for $j = 1, \ldots, N_0$.

Assumption 4.2. It will henceforth be supposed that : (1) Γ is of class $C^{\kappa_0} \cap C^{s_1}$, where $\kappa_0 =$ max$\{t_1,$max$\{\, -\sigma_j \,\}_1^{N_0}\,\}$; (2) for each pair j, k, $a_\alpha^{jk} \in C^{s_j}(\overline{\Omega})$ for $|\alpha| \leq s_j + t_k$ if $s_j > 0$, while if $s_j = 0$, then $a_\alpha^{jk} \in L_\infty(\Omega)$ if $|\alpha| < s_j + t_k$ and $a_\alpha^{jk} \in C(\overline{\Omega})$ for $|\alpha| = s_j + t_k$: (3) for each pair j, k, $b_\alpha^{jk} \in C^{-\sigma_j}(\Gamma)$ for $|\alpha| \leq s_j + t_k$.

In the sequel we shall also require the following notation. For $x \in \overline{\Omega}, \xi \in \mathbb{R}^n$, and $1 \leq r \leq d$, let $\mathcal{A}_{11}^{(r)}(x, \xi) = \left(\overset{o}{A}_{jk}(x, \xi)\right)_{j,k=1}^{k_r}$, while for $x \in \Gamma, \xi \in \mathbb{R}^n$, and $1 \leq \ell_1, \ell \leq d$ we let $\mathcal{B}_{\ell_1}^{(r,\ell)}(x, \xi) =$

$\left(\overset{o}{B}_{jk}\ (x,\xi)\right)_{\substack{j=N_{\ell-1}(1-\delta_{r,\ell})+1,\ldots,N_\ell\\ k=1,\ldots,k_{\ell_1}}}$, where $\delta_{r,\ell}$ denotes the Kronecker delta. In addition we let $\tilde{I}_{1,0} = \tilde{I}_1$ and $\tilde{I}_{r,0} =$ diag$\left(0\tilde{I}_1,\ldots,0\tilde{I}_{r-1},\tilde{I}_r\right)$ for $r = 2,\ldots,d$.

Note that when $x^0 \in \Gamma$ we can rewrite the boundary problem (1.1), (1.2) in terms of the local coordinates at x^0 as explained in Definition 3.4. Then suppoing that this has been done, we shall in the sequel be concerned with the boundary problem

$$\overset{o}{A}\ (0,D)u(x) - \lambda\, u(x) = f(x) \text{ for } x \in \mathbb{R}^n_+,$$
$$\overset{o}{B}_j\ (0,D)u(x) = g_j(x') \text{ at } x_n = 0 \text{ for } j = 1,\ldots,N_0,$$

and corresponding to this boundary problem we define the matrices $\mathcal{A}^{(r)}_{jk}(0,\xi)$, $\mathcal{B}^{(r,\ell)}_{\ell_1}(0,\xi)$ in precisely the same way as their analogues were define in the original coordinate system.

Definition 4.3. Let $\mathcal{L}$ be a closed sector in the complex plane with vertex at the origin. Then the operator $A(x,D) - \lambda\, I_N$ will be called parameter-elliptic in $\mathcal{L}$ if $\det\left(\mathcal{A}^{(r)}_{11}(x,\xi) - \lambda\tilde{I}_{r,0}\right) \neq 0$ for $x \in \overline{\Omega},\ \xi \in \mathbb{R}^n\setminus\{0\}$, and $\lambda \in \mathcal{L},\ r = 1,\ldots,d$.

In the sequel we let $\mathbb{C}_\pm = \{z \in \mathbb{C},\ \text{Im}\, z \gtrless 0\}$.

Definition 4.4. Suppose that the operator $A(x,D) - \lambda\, I_N$ is parameter-elliptic in the sector $\mathcal{L}$ introduced above. Let x^0 be an arbitrary point of Γ and assume that the boundary problem (1.1), (1.2) has been rewritten in a local coordinate system associated with x^0 in the manner just explained. Then the operator $A(x,D) - \lambda\, I_N$ will be called properly parameter-elliptic in $\mathcal{L}$ if the following conditions are satisfied.

(1) The polynomial det$\left(\mathcal{A}^{(r)}_{11}(0,\xi',z) - \lambda\tilde{I}_{r,0}\right)$ has precisely N_r zeros lying in $\mathbb{C}_+$ for $\xi' \in \mathbb{R}^{n-1}\setminus\{0\}$ and $\lambda \in \mathcal{L},\ r = 1,\ldots,d$.

(2) The polynomial det$\left(\mathcal{A}^{(r)}_{11}(0,0,z) - \lambda\tilde{I}_{r,0}\right)$ has precisely $N_r - N_{r-1}$ zeros lying in $\mathbb{C}_+$ for $\lambda \in \mathcal{L}\setminus\{0\},\ \ r = 2,\ldots,d$.

Remark 4.1. Referring to Condition (1) of Definition 4.4, we know from (8, Section 2) that det $\left(\mathcal{A}^{(r)}_{11}(0,\xi',z) - \lambda\tilde{I}_{r,0}\right)$ has precisely N_r zeros in $\mathbb{C}_+$ if $r = 1$ or if $r > 1$ and $n > 2$. In the sequel, when proper parameter-ellipticity is supposed, it will be assumed that this is also the case when $r > 1$ and $n = 2$. Turning next to Condition (2) of the definition, it is clear that the number of zeros of the determinant in $\mathbb{C}_+$ (resp. $\mathbb{C}_-$) does not depend upon λ. Hence it follows from an expansion of the determinant in powers of z and λ that Condition (2) always holds if $\tilde{m}_r$ is even or if $\tilde{m}_r$ is odd, $k_r - k_{r-1}$ is even, and there is a $\lambda \in \mathcal{L}\setminus\{0\}$ such that $-\lambda \in \mathcal{L}$. Lastly we mention at this point that it is also clear from what was said above that Condition (2) is always satisfied if the operator $A(x,D)$ is essentially upper triangular at x^0 (see Definition 4.6 below)

Definition 4.5. Let $\mathcal{L}$ denote the sector introduced in Definition 4.3 above. Then we say that the boundary problem (1.1), (1.2) is parameter-elliptic in $\mathcal{L}$ if $A(x,D) - \lambda\, I_N$ is properly parameter-elliptic in $\mathcal{L}$ and the following conditions are satisfied. Let x^0 be an arbitrary point of Γ and suppose that the boundary problem (1.1), (1.2) has been rewritten in a local coordinate system associated with x^0, as explained above. Then

(1) the boundary problem on the half-line

$$\mathcal{A}^{(r)}_{11}(0,\xi',D_n)v(x_n) - \lambda\tilde{I}_{r,0}v(x_n) = 0 \text{ for } x_n > 0,$$
$$\mathcal{B}^{(r,r)}_{r}(0,\xi',D_n)v(x_n) = 0 \text{ at } x_n = 0,$$
$$|v(x_n)| \to 0 \text{ as } x_n \to \infty, \tag{4.1}$$

has only the trivial solution for $\xi' \in \mathbb{R}^{n-1}\setminus\{0\},\ \lambda \in \mathcal{L}$ and $1 \le r \le d$;

(2) the boundary problem on the half-line

$$\mathcal{A}_{11}^{(\ell)}(0,0,D_n)v(x_n) - \lambda\,\tilde{I}_{\ell,0}v(x_n) = 0 \text{ for } x_n > 0,$$
$$\mathcal{B}_{\ell}^{(r,\ell)}(0,0,D_n)v(x_n) = 0 \text{ at } x_n = 0,$$
$$|v(x_n)| \to 0 \text{ as } x_n \to \infty,$$

has only the trivial solution for $\lambda \in \mathcal{L} \setminus \{0\}$, $1 \le r < d$ and $r < \ell \le d$.

Remark 4.2. Fixing our attention again upon the boundary problem (1.1), (1.2), suppose that there exists the monotonic decreasing sequence of positive integers $\{\,t'_j\,\}_1^N$ such that $s_j = t_j = t'_j$ for $j = 1,\dots,N$ and that the boundary conditions (1.2) are those of Dirichlet. Suppose in addition that $\overset{o}{A}\,(x,\xi)$ is positive definite at each point of $\overline{\Omega} \times S^{n-1}$, where $S^{n-1} = \{\,\xi \in \mathbb{R}^n \,\big|\, |\xi| = 1\,\}$. Lastly suppose that $A(x,D) - \lambda\,I_N$ is properly parameter-elliptic in $\mathcal{L}$ and that $\mathcal{L}$ intersects $\overline{\mathbb{R}_+}$ only at the origin. Then we know from (16) that the boundary problem (1.1), (1.2) is parameter-elliptic in $\mathcal{L}$. On the other hand, if we only suppose that $\overset{o}{A}\,(x,\xi)$ is symmetric at each point of $\overline{\Omega} \times S^{n-1}$ and that $\mathcal{L}$ intersects $\mathbb{R}$ only at the origin, then it was pointed out in (16) that the boundary problem (1.1), (1.2) is parameter- elliptic in $\mathcal{L}$ provided that the boundary problem (4.1) has only the trivial solution when $\lambda = 0$ for $r = 1,\dots,d$.

Definition 4.6. Let $x^0 \in \Gamma$. Then we say that the operator $A(x,D)$ is essentially upper triangular at x^0 if $a_\alpha^{jk}(x^0) = 0$ for $|\alpha| = s_j + t_k$, $k_{\ell-1} < j \le k_\ell$, $1 \le k \le k_{\ell-1}$, $\ell = 2,\dots,d$. Likewise we say that the operator $B(x,D) = (B_{jk}(x,D))_{\substack{j=1,\dots,N_0\\k=1,\dots,N}}$ is essentially upper triangular at x^0 if $b_\alpha^{jk}(x^0) = 0$ for $|\alpha| = \sigma_j + t_k, N_{\ell-1} < j \le N_\ell, 1 \le k \le k_{\ell-1}$, $\ell = 2,\dots,d$.

From (3) we now have the following result. Here, for $1 \le j \le N_0$, we put $\pi(j) = 1$ if $0 < j \le N_1$ and $\pi(j) = r$ if $N_{r-1} < j \le N_r$ and $r > 1$.

Theorem 4.1. *Suppose that the boundary problem* (1.1)*,* (1.2) *is parameter-elliptic in* $\mathcal{L}$*. Suppose also that at least one of the following conditions hold: (1) the boundary conditions* (1.2) *are of Dirichlet type at every point of* Γ*; (2) the operators* $A(x,D)$ *and* $B(x,D)$ *are both essentially upper triangular at every point of* Γ*. Then there exists the constant* $\lambda_0 = \lambda_0(p) > 0$ *such that for* $\lambda \in \mathcal{L}$ *with* $|\lambda| \ge \lambda_0$*, the boundary problem* (1.1)*,* (1.2) *has a unique solution* $u = (u_1,\dots,u_N)^T \in \prod_{j=1}^N W_p^{t_j}(\Omega)$ *for every* $f = (f_1,\dots,f_N)^T \in \prod_{j=1}^N H_p^{-s_j}(\Omega)$ *and* $g = (g_1,\dots,g_{N_0})^T \in \prod_{j=1}^{N_0} W_p^{-\sigma_j - 1/p}(\Gamma)$*, and the a priori estimate*

$$\sum_{j=1}^{N} |||u_j|||_{t_j,p,\Omega}^{(j)} \le C \left(\sum_{j=1}^{N} |||f_j|||_{-s_j,p,\Omega}^{(j)} + \sum_{j=1}^{N_0} |||g_j|||_{-\sigma_j-1/p,p,\Gamma}^{(\pi(j))} \right) \tag{4.2}$$

holds, where the constant C *does not depend upon the* f_j, g_j*, and* λ*.*

Remark 4.3. As was shown in (3) the estimate (4.2) is 2-sided, i.e., an estimate reverse to (4.2) holds.

We return again to the paragraph starting with “Notation” following the proof of Theorem 3.1 and introduce the new spaces

$$W_p^{(t,\lambda)}(\Omega) = \prod_{j=1}^{N} W_{p,\lambda}^{t_j}(\Omega), \qquad W_p^{(t)}(\Omega) = \prod_{j=1}^{N} W_p^{t_j}(\Omega),$$
$$H_p^{(-s,\lambda)}(\Omega) = \prod_{j=1}^{N} H_{p,\lambda}^{-s_j}(\Omega), \qquad H_p^{(-s)}(\Omega) = \prod_{j=1}^{N} H_p^{-s_j}(\Omega).$$

We will also suppose for the remainder of this section that the hypotheses of Theorem 4.1 hold and denote by A_p the operator on $H_p^{(-s)}(\Omega)$ that acts as $A(x, D)$ and has domain

$$D(A_p) = \{ \, u \in W_p^{(t)}(\Omega) \, \big| \, B_j(x, D)u(x) = 0 \text{ on } \Gamma \text{ for } j = 1, \ldots, N_0 \, \}.$$

Then as a consequence of Theoerem 4.1 it follows that if $\lambda \in \mathcal{L}$ with $|\lambda| \geq \lambda_0$ and $D(A_p)$ is equipped with the norm $|||\cdot|||_{W_p^{(t,\lambda)}(\Omega)}$, then the mapping $A_p - \lambda \, I_N : D(A_p) \to H_p^{(-s,\lambda)}(\Omega)$ is an isomorphism and $|||A_p - \lambda \, I_N|||_{D(A_p) \to H_p^{(-s)}(\Omega)} \leq C$, where here, and for the remainder of this section, C denotes the generic constant introduced in the text following (3.2). Thus we see that as an operator on $H_p^{(-s)}(\Omega)$, A_p has an non-empty resolvent set. Furthermore, if we let $R_p(\lambda)$ denote its resolvent, then it follows that for $f \in H_p^{(-s)}(\Omega)$ and $\lambda \in \mathcal{L}$ with $|\lambda| \geq \lambda_0$, we have

$$|||R_p(\lambda)f|||_{W_p^{(t,\lambda)}(\Omega)} \leq C|||f|||_{H_p^{(-s,\lambda)}(\Omega)} \leq C\| \, f \, \|_{H_p^{(-s)}(\Omega)}.$$

In addition we can argue as in the proof of Proposition 3.1 to show that for $\lambda \in \mathcal{L}$ with $|\lambda| \geq \lambda_0$, the mapping $R_p(\lambda) : H_p^{(-s)}(\Omega) \to H_p^{(-s)}(\Omega)$ is compact and $|\lambda| \| \, R_p(\lambda) \, \|_{H_p^{(-s)}(\Omega) \to H_p^{(-s)}(\Omega)} \leq C$.

As a consequence of the foregoing results it follows that A_p is a closed, densely defined operator on $H_p^{(-s)}(\Omega)$ with compact resolvent, and hence a discrete spectrum. Thus for reasons made in the text preceding Assumption 3.5, we are led to make the following assumption.

Assumption 4.7. We henceforth suppose that the sector $\mathcal{L}$ of Theorem 4.1 coincides with the sector $\mathcal{L}_\theta$ defined in the text preceding Assumption 3.5 and that $\mathcal{L}_\theta \cup \tilde{B}_{2\epsilon}(0)$ is contained in the resolvent set of A_p.

Finally, with γ denoting the contour defined in the text preceding Theorem 3.2, we can argue as in the proof of that theorem to obtain the following result.

Theorem 4.2. *Suppose that the hypotheses of Theorem 4.1 as well as Assumption 4.7 hold. Then the boundary problem* (1.1), (1.2) *generates a semigroup of operators on* $H_p^{(-s)}(\Omega)$*, namely* $A_p^{-t} = -(2\,\pi\,i)^{-1} \int_\gamma \lambda^{-t} R_p(\lambda) \, d\lambda$, $t \geq 0$*. Furthermore,* A_p^{-t} *is analytic in the half-plane Re*$\,t > 0$.

5 Conclusion

Fixing our attention firstly upon the mono-order case, we see from Theorem 3.2 that our approach to the semigroup problem has enabled us to give a direct proof that the boundary problem (1.1), (1.2) generates a semigroup of operators acting on the space cited there, and completely avoids the difficulties arising in Dreher's approach to this problem as cited in Section 1. Analogous statements also hold for the boundary problem (3.8), (3.9) (see theorem 3.4. Finally , as shown in Theorem 4.2, our approach to the semigroup problem has allowed us to extend the known results for the homogeneous and mono-order problems to a certain class of parameter-elliptic Douglis-Nirenberg systems of multi-order type.

Competing Interests

The author declares that no competing interests exist.

References

[1] Seeley R. Norms and domains of complex powers of A_B^z. Amer. J. Math. 1971;93:299–309.

[2] Dreher M. Resolvent estimates for Douglis-Nirenberg systems. J. Evol. Equ. 2009;9:829–844.

[3] Denk R. Faierman M. Estimates for solutions of a parameter-elliptic multi-order systmem of differential equations. Integr. Equ. Oper. Theory. 2010;66:327–365.

[4] Grubb G. Kokholm N.J. A global calculus of parameter-dependent pseudodifferential boundary problems in L_p Sobolev spaces. Acta Math. 1993;171:1–100.

[5] Triebel H. Interpolation Theory, Function Spaces, Differential Operators. North-Holland, Amsterdam; 1978.

[6] Agranovich M.S. Denk R. Faierman M. Weakly smooth nonselfadjoint elliptic boundary problems. Math. Top. 1997;14 (1997):138–199.

[7] Grisvard P. Elliptic Problems in Nonsmooth Domains. Pitman, London:1992.

[8] Agranovich M.S. Vishik M.I. Elliptic problems with a parameter and parabolic problems of general form. Russ. Math. Surveys. 1964;19:53–157.

[9] Denk R. Faierman M. Möller M. An elliptic boundary problem for a system involving a discontinuous weight. Manuscripta Math. 2001;108:289–317.

[10] Geymonat G. Grisvard P. Alcuni risultati di teoria spettrale per i problemi ai limiti lineari ellittici. Rend. Sem. Mat. Univ. Padova. 1967;38:127–173.

[11] Volevich L. R. Sovability of boundary value problems for general elliptic systems. Amer. Math. Soc. Transl. 1968;67:182–225.

[12] Krasnoselskii M. A. Zabreiko P. P. Pustylnik E. I. Sobolevskii P. E. Integral Operators in Spaces of Summable Functions. Noordoff, Amsterdam; 1976.

[13] Kato T. Perturbation Theory for linear Operators. Springer, Berlin; 1976. 42.

[14] Agmon S. Lectures on Elliptic Boundary Value Problems. Van Nostrand, Princeton N.J.:1965.

[15] Agmon S. On the eigenfunctions and on the eigenvalues of general elliptic boundary value problems. Comm. Pure Appl. Math. 1962;15:119–147.

[16] Faierman M. Eigenvalue asymptotics for the non-selfadjoint operator induced by a parameter-elliptic multi-order boundary problem. Integr. Equ. Oper. Theory. 2012;74: 25– 42.

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