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Maximum Gap among Integers Having a Common Divisor with an Odd Semi-prime

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Author's contribution

The sole author designed, analysed, interpreted and prepared the manuscript.

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Abstract

For an odd semi-prime $N = pq$ with $p < q < 2p$, this paper demonstrates that the maximum gap between two integers sharing a common divisor with N is $p - 1$. Within interval $[1, N - 1]$ there exists a sequence of such gaps that can be periodically grouped into small clusters determined by the quotient of p divided by $q - p$. Furthermore, the total number of the terms in the sequence is an odd number no smaller than 1. These findings illustrate that the large gaps among multiples of the divisors of a composite odd integer are distributed sparely and periodically. Such distribution is advantageous for designing randomized algorithms capable of identifying a divisor of a composite odd integer within a limited range.

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1 Introduction

This introductory part raises a problem and makes a simple review of its relevant literatures.

1.1 Problems From Observation

Given a semiprime $N = 15$ that has two divisors, 3 and 5; Checking each integer from 3 to 14 knows that integers 3, 6, 9, and 12 are multiples of 3, while integers 5 and 10 are multiples of 5. Using the terminologies in [1] and [2], the multiples of 3 are hosts of the divisor 3 , the multiples of 5 are hosts of the divisor 5, and each of these multiples is a host of N's divisors. By arranging all these hosts in order, a sequence can be achieved.

$$3, 5, 6,|, 9, 10, 12$$

Using the symbol | to express the 'middle' of the sequence, hosts of 3, 3 and 6, and hosts of 5, 5 and 10, are seen symmetrically distributed with respect to |. Using a terminology 'gap' to describe the number of integers between two given integers, it is seen that pairs (5,6) and (10, 9) have gap 0, pairs (3,5) and (12,10) have gap 1, and pair (6,9) has the maximal gap 2. Regarding pair (6,9) is symmetric to itself, pairs of the same gap are symmetrically distributed with respect to |. Changing N to 119, which has two divisors of 7 and 17, leads to the following host sequence

$$7, 14, 17, 21, 28, 34, 35, 42, 49, 51, 56,|, 63, 68,70, 77, 84, 85, 91, 98, 102, 105, 112$$

It can be seen that gap 0 comes from (34, 35) and (85, 84), gap 1 from (49, 51) and (70,68), gap 2 from (14, 17) and (105,102), gap 3 from (17, 21) and (102, 98), gap 4 from (56,51) and (63,68), gap 5 from (28,34) and (91,85), and the maximal gap 6 from (7,14), (21, 28), (35, 42), (42,49), (56,63), (70,77), (84,77), (98,91), and (112,105). Pairs of the same gap are also symmetric with respect to |.

The phenomena stated above were first observed and studied in paper [3]. That paper proved the property of the symmetric distribution of the gaps between two hosts hosting distinct divisors of a semi-prime and the existence of gap 0. This paper continues the study of that paper and shows how the maximum gaps are distributed for the case N is a semi-prime whose divisor ratio is smaller than 2.

The paper consists of six sections. This introductory section raises the observed phenomena and makes a brief on the related literature to show that the problem raised here is truly a new one for which little previous study has been made; section 2 introduces symbols and notations that will be used in later sections; section 3 presents the lemmas, corollaries and theorems proved in this paper; section 4 presents numerical tests; section 5 is the conclusion.

1.2 Simple Review of Relevant Literatures

The topic of this paper is related with two issues in number theory [4]: one is the study of the gaps between integers and the other is the distributions of the divisors of a composite integer. The first one can be traced hundreds of years ago, mainly involved in the exploring the gaps between primes, between integers in an arithmetic progression, and between integers in some particular set of integers. The early researches of the first kind can be found in [5], [6], and [7]; the recent researches can be seen in [8], [9], and [10]. Beath-Brown D R and Iwaniec H in [5] investigated the difference between consecutive primes, Galambos J and Katai I in [6] and [7] researched the gaps in a particular sequence of integers of positive density, Brandon Y Wang and Wang X in [8] proved a symmetrical distribution of primes and their gaps, Melvyn B Nathanson in [9] researched arithmetic progressions contained in sequences with bounded gaps, and Liu Y in [10] estimated bounded gaps between products of distinct primes.

The second issue mainly concerns the distribution of an integer's divisors in an interval or a sequence. Early researches can be found in [11] and [12]; recent ones was summarized in the introductory section of [1]. Jean-Marie De Koninck in [11] studied the distance between consecutive divisors of an integer; Berend D and Harmse J E in [12] reported gaps between consecutive divisors of factorials. Seen in the literatures list [1], the relevant researches have been continued because it is closely related with the study of integer factorization.

The problem raised in this paper concerns the gaps between the integers having a common divisor with a third composite integer. It does not belong to either of the two mentioned issues. It is therefore of a new kind.

2 Terminologies, Symbols and Notations

This section presents necessary symbols, notations, and definitions for later investigation.

2.1 Previously-used Terminologies, Symbols and Notations

This paper continues using symbols and notations introduced in [1], [2], and [3].

2.2 New Terminologies, Symbols and Notations

Let n be an odd integer and $S = \{1, 2, 3, ..., n-1\}$; integer $r \in S$ and $n-r \in S$ are said to be *symmetric modulo n*. In this whole paper, symbol $H^x_{[a,b]}$ means the set of all the hosts of x in interval $[a, b]$ and $H^x_{[a,b]} \cap [c, d]$ means the intersection of $H^x_{[a,b]}$ with the hosts of x in interval $[c, d]$. If $H^x_{[a,b]} \cap [c, d] = \emptyset$, interval $[c, d]$ is called an x-free interval. An integer interval $[h^y_l, h^y_r]$ both of whose two ends are hosts of y is called a y-enclosed interval. Symbol G^x means a gap taking value x and $G^{x \cdot\cdot y}$ means a set of gaps taking values from x to y.

3 Main Results

This section presents main results obtained and proved in this paper. It consists of two subsections: Lemmas and Theorems. Lemmas are fundamental mathematical results proved by primary number theory and Theorems are proven based on the Lemmas to answer the problems raised in the introductory part.

3.1 Lemmas

Lemma 3.1. *Let p and r be positive integers with $0 < r < p$. Then in the case $r = 1$ there is not an integer α with $1 < \alpha < p$ that enables $\alpha r \geq p$; whereas, in the case $r > 1$ there is at least one such α with $1 < \alpha < p$ that enables $\alpha r \geq p$ and $(\alpha - 1)r < p$. Among all those candidates of α, $\alpha = \lceil \frac{p}{r} \rceil$ is the smallest one.*

Proof. The case $r = 1$ is obviously true. The other case, $r \geq 2$, can be proven with proof by contradiction. Assume there is not an α with $1 < \alpha < p$ that makes $\alpha r > p$; then $(p-1)r < p \Leftrightarrow r < 1 + \frac{1}{p-1}$, contradictory to $r \geq 2$. In fact, taking $\alpha = \lceil \frac{p}{r} \rceil$ yields

$$\alpha r = \left\lceil \frac{p}{r} \right\rceil r \geq p$$

and

$$\frac{p}{r} \leq \alpha = \left\lceil \frac{p}{r} \right\rceil < \frac{p}{r} + 1 \Leftrightarrow p - r \leq (\alpha - 1)r < p.$$

Now assume $\beta = \alpha - \delta$, $\beta r > p$, and $(\beta - 1)r < p$, where $\delta \geq 1$ is an integer; then

$$(\left\lceil \frac{p}{r} \right\rceil - \delta)r \geq p \Leftrightarrow \left\lceil \frac{p}{r} \right\rceil - \frac{p}{r} \geq \delta$$

By (P19) in [13], $\lceil \frac{p}{r} \rceil = \lfloor \frac{p-1}{r} \rfloor + 1$, yielding

$$(\left\lceil \frac{p}{r} \right\rceil - \delta)r \geq p \Leftrightarrow \left\lceil \frac{p}{r} \right\rceil - \frac{p}{r} \geq \delta \Leftrightarrow \left\lfloor \frac{p-1}{r} \right\rfloor - \frac{p-1}{r} \geq \delta - 1 + \frac{1}{r} > 0$$

contradictory to the fact $\lfloor \frac{p-1}{r} \rfloor \leq \frac{p-1}{r}$.

Hence the lemma consequently holds. □

Lemma 3.2. *Let p and q be two odd integers with $(p, q) = 1$, $q = p + r$, and $1 < r < p$; then there exists an integer α with $1 < \alpha \leq p - 1$ that enables $(\alpha - 1)r < p$, $\alpha r > p$,*

$$(\alpha - 1)p < (\alpha - 1)q < \alpha p \tag{3.1}$$

and

$$(\alpha + 1)p < \alpha q < (\alpha + 2)p. \tag{3.2}$$

Proof. Since p and q are odd, r is even, namely, $r \geq 2$. Lemma 3.1 ensures the existence of α satisfying $1 < \alpha \leq p - 1$, $(\alpha - 1)r < p$, and $\alpha r > p$. Next prove it also satisfies (3.1) and (3.2). The condition $(p, q) = 1$ yields $(p, r) = 1$. By $\alpha r > p$, let $\alpha r = sp + t$ with $s \geq 1$ and $0 < t < p$ being integers; then

$$(\alpha - 1)q = (\alpha - 1)p + (\alpha - 1)r, 0 < (\alpha - 1)r < p \tag{3.3}$$

and

$$\alpha q = \alpha p + \alpha r = (\alpha + s)p + t, 0 < t < p. \tag{3.4}$$

From (3.3), (3.1) surely holds. Next prove $s = 1$. In fact $\alpha r = sp + t$ yields

$$(\alpha - 1)r = sp + t - r \tag{3.5}$$

Since $0 < t < p$ and $1 < r < p$, it is known

$$-(p - 2) \leq t - r \leq p - 3 < p - 2,$$

indicating by (3.5)

$$(\alpha - 1)r \geq sp - p + 2$$

If $s > 1 \Leftrightarrow s \geq 2$, it derives $(\alpha - 1)r \geq p + 2$, which is contradictory to $0 < (\alpha - 1)r < p$. Accordingly, (3.4) becomes $\alpha q = (\alpha + 1)p + t$ with $0 < t < p$, which is identical to (3.2). □

3.2 Corollaries and Theorems

Corollary 3.3. *Let $N = pq$ be an odd integer and $I_N = [1, N - 1]$ be an integer interval, where p and q are odd integers with $1 < p < q$ and $(p, q) = 1$; Assume $h^p \in I_N$ and $h^q \in I_N$ are hosts of p and q, respectively; then*

$$0 \leq g_{h^p}^{h^q} \leq p - 2.$$

Proof. The hosts of p and q in I_N are given by

$$p, 2p, 3p, ..., (\frac{q-1}{2})p, (\frac{q+1}{2})p, ..., (q - 1)p \tag{3.6}$$

and

$$q, 2q, 3q, ..., (\frac{p-1}{2})q, (\frac{p+1}{2})q, ..., (p - 1)q. \tag{3.7}$$

Since $(q - 1)p - (p - 1)q = q - p > 0$, αq with integer $1 \leq \alpha \leq p - 1$ lies in the integer interval $[1, (q - 1)p]$. By $(p, q) = 1$, αq must lie between two adjacent hosts of p, indicating $0 \leq g_{h^p}^{h^q} \leq p - 2$ because the gap between two adjacent hosts of p is $p - 1$. □

Corollary 3.4. *Let p and q be odd integers with $1 < p < q$; then there is not a host of q between $(\frac{q-1}{2})p$ and $(\frac{q+1}{2})p$. There are at least two hosts of p between $(\frac{p-1}{2})q$ and $(\frac{p+1}{2})q$; particularly, $(\frac{q-1}{2})p$ and $(\frac{q+1}{2})p$ are exact two hosts of p between $(\frac{p-1}{2})q$ and $(\frac{p+1}{2})q$ if $1 < \frac{q}{p} < 2$.*

Proof. The first conclusion can be proved using proof by contradiction. A host of q must be of the form αq with $\alpha \geq 1$. Assume there is such an α that makes $(\frac{q-1}{2})p < \alpha q < (\frac{q+1}{2})p$. Then it follows

$$(q-1)p < 2\alpha q < (q+1)p \Leftrightarrow p - \frac{p}{q} < 2\alpha < p + \frac{p}{q} \Rightarrow p = 2\alpha$$

leading to a contradiction to that p is odd.

To prove the second conclusion, consider an integer α that satisfies

$$(\frac{p-1}{2})q < \alpha p < (\frac{p+1}{2})q. \tag{3.8}$$

Then it follows

$$q - \frac{q}{p} < 2\alpha < q + \frac{q}{p}$$

Note that, $q - \left\lfloor \frac{q}{p} \right\rfloor, q - \left\lfloor \frac{q}{p} \right\rfloor + 1, ..., q, ..., q + \left\lfloor \frac{q}{p} \right\rfloor - 1$, and $q + \left\lfloor \frac{q}{p} \right\rfloor$ are integers between $q - \frac{q}{p}$ and $q + \frac{q}{p}$, meaning α can take at least two values to hold (3.8). In the case $1 < \frac{q}{p} < 2 \Leftrightarrow \left\lfloor \frac{q}{p} \right\rfloor = 1$, $q-1$, q, and $q+1$ are three integers to hold (3.8), meaning $\alpha = \frac{q-1}{2}$ and $\alpha = \frac{q+1}{2}$ are the only two integers to hold (3.8). □

Theorem 3.5. *Let $N = pq$ be an odd integer and $I_N = [1, N-1]$ be an integer interval, where p and q are odd integers such that $1 < p < q$ and $(p, q) = 1$; then the maximal gap between two adjacent hosts of N's divisors in I_N is $p-1$, and there is always such a gap in the middle of I_N.*

Proof. By Corollary 3.3, gaps between hosts of p and hosts of q are between 0 and $p-2$. By Corollary 3.4, there are at least two hosts of p between $(\frac{p-1}{2})q$ and $(\frac{p+1}{2})q$. Because the gap between arbitrary two adjacent hosts of p is $p-1$, the theorem certainly holds. □

Theorem 3.6. *Given an odd integer $N = pq$ whose divisors p and q are odd integers satisfying $(p, q) = 1$, $q = p + r$ with $1 < r < p$; let $\omega = \lceil \frac{p}{r} \rceil$,$\varsigma = \lfloor \frac{p+1}{2\omega} \rfloor - 1$, and integer intervals be given by*

$$\begin{aligned}
&I_N = [1, N-1],\\
&I_0 = [q, (\omega-1)q],\\
&I_1 = [\omega q, (2\omega-1)q],\\
&...,\\
&I_k = [k\omega q, ((k+1)\omega-1)q],\\
&...,\\
&I_\varsigma = [\varsigma\omega q, ((\varsigma+1)\omega-1)q],\\
&I_{\varsigma+1} = [(\varsigma+1)\omega q, (\tfrac{p-1}{2})q].
\end{aligned}$$

Then for $0 \leq j \leq \varsigma$, G^{p-1} exists between the end of I_j and the start of I_{j+1} except for the one near and out of the end of $I_{\varsigma+1}$. There are at least $2\varsigma + 1$such gaps distributed symmetrically in I_N.

Proof. The given conditions show that r is an even integer satisfying $2 \leq r \leq p-1$. For convenience of later reasoning, let $r_0 = r$ and $\omega_0 = \omega$; then $q = p + r_0$, $(p, r_0) = 1$, and $(q, r_0) = 1$. Consider the following sequence (3.9) given by

$$q = p + r_0, 2q = 2p + 2r_0, \quad ..., \alpha q = \alpha p + \alpha r_0, ..., (p-1)q = (p-1)p + (p-1)r_0 \tag{3.9}$$

By Lemmas 3.1 and 3.2 , $\omega_0 = \left\lceil \frac{p}{r_0} \right\rceil$ is the smallest integer that makes $(\omega_0 - 1)r_0 < p$, $\omega_0 r_0 > p$, $(\omega_0 - 1)p < (\omega_0 - 1)q < \omega_0 p$, and $(\omega_0 + 1)p < \omega_0 q < (\omega_0 + 2)p$, meaning

(i). $(\omega_0 - 1)r_0 < p \Rightarrow (\omega_0 - j)r_0 < p$ and $H^q_{[1,N]} \cap [jp, (j+1)p] \neq \emptyset$, where integer j satisfies $1 \leq j \leq \omega_0 - 1$.

(ii). There is not a host of q between $\omega_0 p$ and $(\omega_0 + 1)p$, namely,

$$H^q_{[1,N]} \cap [\omega_0 p, (\omega_0 + 1)p] = \emptyset. \tag{3.10}$$

Now taking half of the ordered sequence (3.9) obtains

$$q = p + r_0, 2q = 2p + 2r_0, \quad ..., \alpha q = \alpha p + \alpha r_0, ..., (\frac{p-1}{2})q = (\frac{p-1}{2})p + (\frac{p-1}{2})r_0 \tag{3.11}$$

Let

$$G_1 = \{q, 2q, ..., (\omega_0 - 1)q\}$$

and

$$G_2 = \{\omega_0 q, (\omega_0 + 1)q, ..., \frac{p-1}{2}q\}.$$

Then the number of elements in G_2 is calculated by

$$M = \frac{p-1}{2} - \omega_0 + 1 = \frac{p+1}{2} - \omega_0 \tag{3.12}$$

Obviously, G_2 is empty in the case $M = 0 \Leftrightarrow \omega_0 = \frac{p+1}{2}$. This time the last element of G_1 is $(\omega_0 - 1)q = \frac{p-1}{2}q$. Referring to Corollary 3.4, G^{p-1} is known to occur once between $\frac{p-1}{2}q$ and $\frac{p+1}{2}q$ in (3.9), validating the theorem.

If $M > 0$, the proved (i) and (3.10) show that G^{p-1} does not occur within G_1 but occurs once between the last element of G_1 and the first element of G_2. Now investigate the situation in G_2. Let $\omega_0 r_0 = p + r_1$ with $0 < r_1 < p$; then $r_1 = \omega_0 r_0 - p$, leading to

$$\omega_0 q = (\omega_0 + 1)p + r_1, 0 < r_1 < p. \tag{3.13}$$

and the following calculations,

$$\begin{aligned}
&(\omega_0 + 1)q = \omega_0 q + q = ((\omega_0 + 1) + 1)p + r_1 + r_0, \\
&(\omega_0 + 2)q = (\omega_0 + 1)q + q = ((\omega_0 + 1) + 2)p + r_1 + 2r_0, \\
&(\omega_0 + 3)q = (\omega_0 + 2)q + q = ((\omega_0 + 1) + 3)p + r_1 + 3r_0, \\
&\cdots\cdots.
\end{aligned}$$

The general formula for the above calculations is easily derived by

$$(\omega_0 + j)q = (\omega_0 + 1 + j)p + r_1 + jr_0. \tag{3.14}$$

where $0 \leq j \leq M - 1$ is an integers.

Substituting $j + 1$ for$j$ in (3.14) results in the adjacent follow-up of $(\omega_0 + j)q$ by

$$(\omega_0 + j + 1)q = (\omega_0 + 1 + j + 1)p + r_1 + (j + 1)r_0. \tag{3.15}$$

In the case $\omega_0 < M - 1$, taking $j = \omega_0 - 1$ in (3.14) and (3.15) leads to, respectively,

$$(2\omega_0 - 1)q = 2\omega_0 p + r_1 + (\omega_0 - 1)r_0 \tag{3.16}$$

and

$$2\omega_0 q = (2\omega_0 + 1)p + r_1 + \omega_0 r_0 \tag{3.17}$$

With $\omega_0 r_0 = p + r_1$, (3.17) is turned to be

$$2\omega_0 q = 2(\omega_0 + 1)p + 2r_1. \tag{3.18}$$

Then it follows

$$(2\omega_0 + 1)q = (2(\omega_0 + 1) + 1)p + 2r_1 + r_0 \tag{3.19}$$

Seen from (3.13), (3.16), (3.18), and (3.19), calculations of the elements in G_2 reveal an ω_0-periodic phenomenon. A positive integer k can always correspond to ω_0 calculations starting with $k\omega_0 q = k(\omega_0+1)p + kr_1$ and ending with $((k+1)\omega_0 - 1)q = ((k+1)\omega_0 + k - 1)p + kr_1 + (\omega_0 - 1)r$, except the last several ones. Accordingly, G_2 can be grouped into $m+1$ small groups in terms of such periodic trait by,

$$G_2 = \{\underbrace{\omega_0 q ..., (2\omega_0 - 1)q}_{group \quad 1}, ..., \underbrace{k\omega_0 q, (k\omega_0 + 1)q, ..., ((k+1)\omega_0 - 1)q}_{group \quad k}, ..., \underbrace{(m+1)q, ..., (p-1)q/2}_{group \quad m+1}\}.$$

where

$$m = \left\lfloor \frac{M}{\omega_0} \right\rfloor = \left\lfloor \frac{p+1}{2\omega_0} \right\rfloor - 1 = \varsigma \tag{3.20}$$

So that

$$m \leq \left\lfloor \frac{r_0}{2} \right\rfloor - 1. \tag{3.21}$$

Each of the first m small groups contains ω_0 elements while the last one contains $M - m\omega_0 = \frac{p+1}{2} - m\omega_0 - \omega_0 = \frac{p+1}{2} - (m+1)\omega_0$ ones.

The j-th member in the k-th group is given by

$$(k\omega_0 + j)q = (k(\omega_0 + 1) + j)p + kr_1 + jr_0. \tag{3.22}$$

where $0 \leq j \leq \omega_0 - 1$for $1 \leq k \leq m$ while $0 \leq j \leq \frac{p+1}{2} - (m+1)\omega_0$ for $k = m+1$.

The last one of group k with $1 \leq k \leq m$ is

$$((k+1)\omega_0 - 1)q = ((k+1)(\omega_0 + 1) - 2)p + kr_1 + (\omega_0 - 1)r_0 \tag{3.23}$$

and the first one of group k+1 with $1 < k \leq m+1$ is

$$(k+1)\omega_0 q = (k+1)(\omega_0 + 1)p + (k+1)r_1. \tag{3.24}$$

For convenience, use q_{k,ω_0-1} for $((k+1)\omega_0 - 1)q$ and R_{k,ω_0-1} for $kr_1 + (\omega_0 - 1)r_0$ in (3.23), $q_{k+1,0}$ for $(k+1)\omega_0 q$ and $R_{k+1,0}$ for $(k+1)r_1$ in (3.24). Rewrite respectively (3.23) and (3.24) to be

$$q_{k,\omega_0-1} = ((k+1)(\omega_0 + 1) - 2)p + R_{k,\omega_0-1}, \tag{3.25}$$

and

$$q_{k+1,0} = (k+1)(\omega_0 + 1)p + R_{k+1,0}. \tag{3.26}$$

Direct calculations show

$$R_{k+1,0} - R_{k,\omega_0-1} = r_0 - p. \tag{3.27}$$

With (3.25) and (3.27), it can be proven that q_{k,ω_0-1} is around $((k+1)\omega_0 + k)p$. In fact, by (3.27) it follows

$$R_{k,\omega_0-1} = R_{k+1,0} + p - r_0 = (k+1)(\omega_0 r_0 - p) + p - r_0.$$

Because $(p, r_0) = 1$, it is sure $p \leq \omega_0 r_0 < p + r_0 \Rightarrow \omega_0 r_0 > p$, and thus

$$p < \omega_0 r_0 < p + r_0 \Leftrightarrow p + 1 \leq \omega_0 r_0 \leq p + r_0 - 1.$$

Hence R_{k,ω_0-1} is bounded by

$$k+1+p-r_0 \leq R_{k,\omega_0-1} \leq kr_0+k+1+p. \tag{3.28}$$

Combining with (3.25) results in

$$((k+1)\omega_0+k)p+k+1-r_0 \leq q_{k,\omega_0-1} \leq ((k+1)\omega_0+k)p+kr_0+k+1. \tag{3.29}$$

Since $1 \leq k \leq m$, by (3.21) $q_{k,\omega_0-1}$is surely around $((k+1)\omega_0+k)p$.

Now look at the integer intervals around $((k+1)\omega_0+k)p$. Let

$$I_{ll} = [((k+1)\omega_0+k-2)p, ((k+1)\omega_0+k-1)p], \tag{3.30}$$

$$I_{lr} = [((k+1)\omega_0+k-1)p, ((k+1)\omega_0+k)p], \tag{3.31}$$

$$I_{rl} = [((k+1)\omega_0+k)p, ((k+1)\omega_0+k+1)p]; \tag{3.32}$$

and

$$I_{rr} = [((k+1)\omega_0+k+1)p, ((k+1)\omega_0+k+2)p], \tag{3.33}$$

These intervals are obviously around $((k+1)\omega_0+k)p$, as illustrated with Fig. 1.

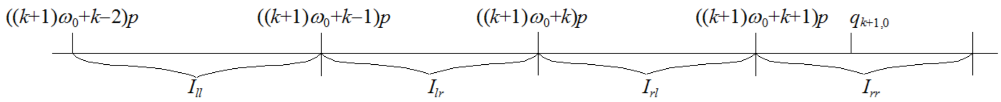

Fig. 1. Intervals are around $((k+1)\omega_0+k)p$

By (3.26), it follows

$$q_{k+1,0} > (k+1)(\omega_0+1)p = ((k+1)\omega_0+k+1)p$$

indicating $q_{k+1,0}$ falls into I_{rr} or an interval right to I_{rr}, as also illustrated with Fig. 1.

It can be proven that q_{k,ω_0-1} cannot be in I_{ll}. In fact, assume $q_{k,\omega_0-1} \in I_l$; then $(p,q)=1$ leads to

$$q_{k,\omega_0-1} \in (((k+1)\omega_0+k-2)p, ((k+1)\omega_0+k-1)p) \tag{3.34}$$

By (3.26),

$$q_{k+1,0} = (k+1)(\omega_0+1)p+(k+1)r_1 = (k+1)\omega_0 p+(k+1)\omega_0 r_0.$$

Hence the smallest gap between $q_{k+1,0}$ and an integer in I_l is

$$q_{k+1,0} - ((k+1)\omega_0+k-1)p = (k+1)(\omega_0 r_0 - p) + 2p > 2p,$$

which is contradictory to $q_{k+1,0} - q_{k,\omega_0-1} = q < 2p$.

Therefore, q_{k,ω_0-1} can lie in three possible intervals: I_{lr}, I_{rl}, and I_{rr}. Next show that either case leaves G^{p-1} to occur.

First, q_{k,ω_0-1} lying in I_{lr} surely leaves G^{p-1} to occur because this time interval I_{rl} is q-free for the reason that $q_{k+1,0}$ lies in I_{rr} or an interval right to I_{rr}.

Now assume $q_{k,\omega_0-1} \in I_{rl}$. Then by (3.25) and (3.32)

$$((k+1)(\omega_0+1)-2)p + R_{k,\omega_0-1} > ((k+1)\omega_0+k)p \Rightarrow R_{k,\omega_0-1} > 2p \Rightarrow R_{k+1,0} > p + r_0$$
$$\Rightarrow q_{k+1,0} = (k+1)(\omega_0+1)p + R_{k+1,0} > ((k+1)\omega_0+k+2)p + r_0$$

saying this time $q_{k+1,0}$ lies in an interval right to I_{rr} and G^{p-1} occurs in I_{rr}.

Likewise, in the case q_{k,ω_0-1} lies in I_{rr}, it holds by (3.25) and (3.33)

$$((k+1)(\omega_0+1)-2)p + R_{k,\omega_0-1} > ((k+1)\omega_0+k+1)p \Rightarrow R_{k,\omega_0-1} > 3p \Rightarrow R_{k+1,0} > 2p + r_0$$
$$\Rightarrow q_{k+1,0} = (k+1)(\omega_0+1)p + R_{k+1,0} > ((k+1)\omega_0+k+3)p + r_0$$

saying this time G^{p-1} occurs in the interval right to I_{rr}.

Therefore every small group k with $1 \le k \le m$ leaves G^{p-1} to occur near and out of its end because q_{k,ω_0-1} is the last member of the group k. For the group $m+1$, G^{p-1} is sure to occur out of its end because its last member is $(p-1)q/2$ which is proven in Corollary 3.4. In the end, the symmetrical property of the hosts in I_N finishes validating the theorem. □

4 Numerical Tests

Results of Theorems 3.5 and 3.6 can be easily tested. Here take $N = 187 = 11 \times 17$ and $N = 713 = 23 \times 31$ as examples to show the numerical tests. Readers can find more examples as well as Maple programs in [14].

Example 1. $N = 187 = 11 \times 17 \Rightarrow p = 11, q = 17, r = 6$. Calculate $\omega_0 = \lceil \frac{p}{r} \rceil = 2$ and $m = \left\lfloor \frac{p+1}{2\omega_0} \right\rfloor - 1 = 2$. By Theorem 2, each of the intervals $[((\omega-1)q, \omega q] = [17, 34]$, $[((2\omega-1)q, 2\omega q] = [51, 68]$, and $[(\frac{p-1}{2})q, (\frac{p+1}{2})q] = [85, 102]$ contains gap $p-1 = 10$. In fact, $[22, 33] \subset [17, 34]$, $[55, 66] \subset [51, 68]$, and $[88, 99] \subset [85, 102]$ are 3 intervals having gap 10. Programmed and drawn with Maple software, Fig. 2. exactly describes the results including their symmetric property.

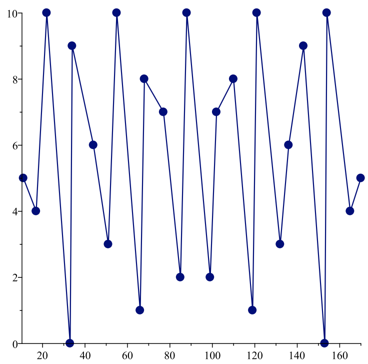

Fig. 2. Distribution of gaps form hosts of 11 and 17

Example 2. $N = 713 = 23 \times 31 \Rightarrow p = 23, q = 31, r = 8$. Calculate $\omega_0 = \lceil \frac{p}{r} \rceil = 3$ and $m = \left\lfloor \frac{p+1}{2\omega_0} \right\rfloor - 1 = 3$. By Theorem 2, each of the intervals $[((\omega_0-1)q, \omega_0 q] = [62, 93]$, $[((2\omega_0-1)q, 2\omega_0 q] = [155, 186]$, $[((3\omega_0-1)q, 3\omega_0 q] = [248, 279]$, and $[(\frac{p-1}{2})q, (\frac{p+1}{2})q] = [341, 372]$ contains gap $p-1 = 22$. In fact, $[69, 92] \subset [62, 93]$, $[161, 184] \subset [155, 186]$, $[253, 276] \subset [248, 279]$, and $[345, 368] \subset [341, 372]$ are 4 intervals having gap 22. Programmed and drawn with Maple software, Fig. 3 exactly describes the results including their symmetric property.

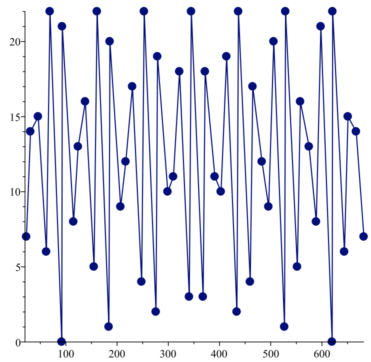

Fig. 3. Distribution of gaps form hosts of 23 and 31

5 Conclusion

Knowing the distribution of the gaps among hosts of the divisors of a composite integer is helpful to design randomized algorithm to search a divisor of The theorems proved in this paper reveal a new symmetric characteristic of the hosts of composite integer. The conclusions in Theorems 1 and 2 indicate that large gaps are distributed periodically and symmetrically. Such a distribution is beneficial for finding a small range to identifying certain expected divisors of un-factorized composite integer.

Nevertheless, readers can see from the numerical experiments provided in [14], there are other large gaps whose distributions are not revealed in this paper. This forms the future research work. Hope the perfect result come soon.

Disclaimer (Artificial Intelligence)

Author(s) hereby declare that NO generative AI technologies such as Large Language Models (ChatGPT, COPILOT, etc) and text-to-image generators have been used during writing or editing of this manuscript.

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Competing Interests

Author has declared that no competing interests exist.

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