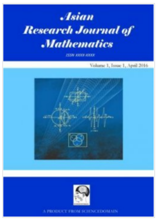

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Boundary Value Method for Numerically Solving Fifth-order Boundary Value Problems

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Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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Abstract

In this work, fifth-order Boundary-Value Problems (BVPs) in Ordinary Differential Equation are solved numerically using Boundary Value Method. Continuous linear multistep methods are developed with continuous coefficients. This constitutes appropriate methods termed the main and additional methods, which are applied sequentially in blocks to approximate the solution. The method is shown to be flexible in handling linear and nonlinear fifth order BVPs. The convergence of the method is discussed. Several numerical examples are shown to illustrate the superiority of the method developed as the approximate solutions derived from the method are compared to the exact solutions of the problem, and other methods from existing literature.

Keywords: Fifth-order boundary value problem; block methods; linear multiste.

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1 Introduction

In this paper, the numerical solution for the fifth-order boundary value problems of the type:

$$\begin{array}{l} y^{(v)} = f(x, y, y', y'', y''', y^{(iv)}), \quad a \leq x \leq b \\ y(a) = \alpha_0, \;\; y'(a) = \alpha_1, \quad y''(a) = \alpha_2, \quad y(b) = \beta_0, \quad y'(b) = \beta_1 \end{array} \tag{1.1}$$

is considered, where α_i, β_j, $(i = 0, 1, 2, \;\; j = 0, 1)$ are finite real arbitrary constants while $f(x)$ is a continuous function defined on the interval $x \in [a, b]$.

Higher order boundary value problems occur in sciences, for example, in the modeling of induction motors in current electricity. Other applications of this type of boundary value problems arise in the mathematical modeling of the viscoelastic flows and other branches of mathematical, physical and engineering sciences. Several numerical methods for solution of (1.1) exist in the literature, and some are specially designed to solve fifth order ordinary differential equations. Fyfe [1] developed quintic polynomial spline functions for the solution of special type of fifth-order boundary-value problems. Caglar et al. [2] developed numerical solution of fifth-order boundary-value problems with sixth-degree B-spline functions, as did [3] who presented a class of methods based on non-polynomial sextic spline functions for the solution of a special fifth-order boundary-value problems. Finite difference solutions were provided by Khan [4]. Many researchers [5, 6] presented approximate analytical solutions of fifth-order boundary value problems by the variational iteration method. Xueqin and Minggen [7] provided an algorithm to solve general linear fifth-order boundary-value problems in the reproducing kernel space $W_2^6[a, b]$. In this paper, solution to fifth order boundary value problems of the type (1.1) using boundary value methods is proposed. The boundary value method has been extensively studied by several researchers and have only used the BVMs to solve initial value problems. see [8], [9], [10], [11], [12], [13] and [14, 15].

2 Derivation of the Method

In this section, a 5-step boundary value method for the solution of fifth-order boundary value problems with appropriate conditions on the interval from x_n to x_{n+5} is developed. It is initially assumed that the solution on the interval $[x_n, x_{n+5}]$ is locally approximated by polynomial of the form,

$$y(x) = \sum_{j=0}^{11} a_j x^j \tag{2.1}$$

which also yield

$$y^{(v)}(x) = \sum_{j=5}^{11} j(j-1)(j-1)(j-3)(j-4) a_j x^{j-5} \tag{2.2}$$

where a_j are unknown coefficients to be determined. Interpolating (2.1) at the points x_{n+i}, $i = 0(1)4$ and collocating (2.2) at the points x_{n+i}, $i = 0(1)5$ gives the following interpolation/collocation equations;

$$
\begin{aligned}
&a_0 + a_1 x_n + a_2 x_n^2 + a_3 x_n^3 + \cdots + a_{11} x_n^{11} - y_n = 0 \\
&a_0 + a_1 x_{n+1} + a_2 x_{n+1}^2 + a_3 x_{n+1}^3 + \cdots + a_{11} x_{n+1}^{11} - y_{n+1} = 0 \\
&a_0 + a_1 x_{n+2} + a_2 x_{n+2}^2 + a_3 x_{n+2}^3 + \cdots + a_{11} x_{n+2}^{11} - y_{n+2} = 0 \\
&a_0 + a_1 x_{n+3} + a_2 x_{n+3}^2 + a_3 x_{n+3}^3 + \cdots + a_{11} x_{n+3}^{11} - y_{n+3} = 0 \\
&a_0 + a_1 x_{n+4} + a_2 x_{n+4}^2 + a_3 x_{n+4}^3 + \cdots + a_{11} x_{n+4}^{11} - y_{n+4} = 0 \\
&120a_5 + 720a_6 x_n + 2520a_7 x_n^2 + 6720a_8 x_n^3 + \cdots + 55440a_{11} x_n^6 - f_n = 0 \\
&120a_5 + 720a_6 x_{n+1} + 2520a_7 x_{n+1}^2 + 6720a_8 x_{n+1}^3 + \cdots + 55440a_{11} x_{n+1}^6 - f_{n+1} = 0 \\
&120a_5 + 720a_6 x_{n+2} + 2520a_7 x_{n+2}^2 + 6720a_8 x_{n+2}^3 + \cdots + 55440a_{11} x_{n+2}^6 - f_{n+2} = 0 \\
&120a_5 + 720a_6 x_{n+3} + 2520a_7 x_{n+3}^2 + 6720a_8 x_{n+3}^3 + \cdots + 55440a_{11} x_{n+3}^6 - f_{n+3} = 0 \\
&120a_5 + 720a_6 x_{n+4} + 2520a_7 x_{n+4}^2 + 6720a_8 x_{n+4}^3 + \cdots + 55440a_{11} x_{n+4}^6 - f_{n+4} = 0 \\
&120a_5 + 720a_6 x_{n+5} + 2520a_7 x_{n+5}^2 + 6720a_8 x_{n+5}^3 + \cdots + 55440a_{11} x_{n+5}^6 - f_{n+5} = 0
\end{aligned}
$$

$$. \tag{2.3}$$

which translates into the matrix equation $A\underline{x} = \underline{b}$, where;

$$
A = \begin{pmatrix}
1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & \cdots & x_n^{11} \\
1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 & x_{n+1}^6 & \cdots & x_{n+1}^{11} \\
\vdots & & & & & & & & \\
1 & x_{n+4} & x_{n+4}^2 & x_{n+4}^3 & x_{n+4}^4 & x_{n+4}^5 & x_{n+4}^6 & \cdots & x_{n+4}^{11} \\
0 & 0 & 0 & 0 & 0 & 120 & 720x_n & \cdots & 55440x_n^6 \\
0 & 0 & 0 & 0 & 0 & 120 & 720x_{n+1} & \cdots & 55440x_{n+1}^6 \\
\vdots & & & & & & & & \\
0 & 0 & 0 & 0 & 0 & 120 & 720x_{n+5} & \cdots & 55440x_{n+5}^6
\end{pmatrix}; \quad
\underline{x} = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{10} \\ a_{11} \end{pmatrix}; \quad
\underline{b} = \begin{pmatrix} y_n \\ y_{n+1} \\ \vdots \\ y_{n+4} \\ f_n \\ f_{n+1} \\ \vdots \\ f_{n+5} \end{pmatrix}
$$

Let the basis function space be given as Using the Gaussian elimination method, we obtained the values of the a_j's, $j = 0(1)11$. Then substituting all the a_j's into (2.1), and after some simplifications, we obtain the continuous linear multistep method of the form

$$
Y(x) = \sum_{i=0}^{4} \alpha_i(x) y_{n+i} + h^4 \sum_{i=0}^{5} \beta_i(x) f_{n+i} \tag{2.4}
$$

where α and β are continuous function of x, $y_{n+i} = y(x_n + ih)$, $f_{n+i} = y^{(v)}(x_n + ih)$. Note that $f_{n+i} = f(x_{n+i}, y_{n+i}, y'_{n+i}, \ldots, y_{n+i}^{(iv)})$, $i = 0, \ldots, 5$. The continuous coefficients $\alpha_i(x)$, $\beta_i(x)$, expressed as functions of t obtained are given below;

$$
\begin{aligned}
&\alpha_0 = \tfrac{1}{24}(6t + 11t^2 + 6t^3 + t^4), \quad \alpha_1 = \tfrac{1}{6}(-8t - 14t^2 - 7t^3 - t^4), \quad \alpha_2 = \tfrac{1}{4}(12t + 19t^2 + 8t^3 + t^4) \\
&\alpha_3 = \tfrac{1}{6}(-24t - 26t^2 - 9t^3 - t^4), \quad \alpha_4 = \tfrac{1}{24}(24 + 50t + 35t^2 + 10t^3 + t^4) \\
&\beta_0 = \tfrac{1}{7257600}(720t + 948t^2 - 820t^3 - 1405t^4 + 504t^6 + 120t^7 - 45t^8 - 20t^9 - 2t^{10}) \\
&\beta_1 = \tfrac{1}{1451520}(14256t + 26460t^2 + 16104t^3 + 4387t^4 - 672t^6 - 144t^7 + 63t^8 + 24t^9 + 2t^{10}) \\
&\beta_2 = \tfrac{1}{725760}(81648t + 153084t^2 + 84532t^3 + 12329t^4 + 1008t^6 + 168t^7 - 99t^8 - 28t^9 - 2t^{10}) \\
&\beta_3 = \tfrac{1}{725760}(57168t + 133764t^2 + 109168t^3 + 34417t^4 - 2016t^6 - 48t^7 + 153t^8 + 32t^9 + 2t^{10}) \\
&\beta_4 = \tfrac{1}{1451520}(-2160t + 4068t^2 + 20700t^3 + 24215t^4 + 12096t^5 + 2184t^6 - 360t^7 - 225t^8 - 36t^9 - 2t^{10}) \\
&\beta_5 = \tfrac{1}{7257600}(2160t + 1932t^2 - 3400t^3 - 4265t^4 + 2016t^6 + 1200t^7 + 315t^8 + 40t^9 + 2t^{10})
\end{aligned}
$$

where $t = \dfrac{x - x_{n+4}}{h}$. Evaluating $Y(x)$ and the derivatives $Y^{(\mu)}(x)$, $\mu = 1(1)4$ in (2.4) at the point x_{n+5} and after some algebraic simplification, the following discrete 5-step LMM are obtained as the main method;

$$
\begin{cases}
y_{n+5} = -y_n + 5y_{n+1} - 10y_{n+2} + 10y_{n+3} - 5y_{n+4} + \frac{1}{24}h^5\left(f_{n+1} + 11f_{n+2} + 11f_{n+3} + f_{n+4}\right) \\
hy'_{n+5} = \frac{25y_n}{12} - \frac{61y_{n+1}}{6} + \frac{39y_{n+2}}{2} - \frac{107y_{n+3}}{6} + \frac{77y_{n+4}}{12} \\
-\frac{h^5}{10080}\left(3f_n - 894f_{n+1} - 9679f_{n+2} - 10819f_{n+3} - 1624f_{n+4} - 3f_{n+5}\right) \\
h^2y''_{n+5} = \frac{35y_n}{12} - \frac{41y_{n+1}}{3} + \frac{49y_{n+2}}{2} - \frac{59y_{n+3}}{3} + \frac{71y_{n+4}}{12} \\
-\frac{h^5}{302400}\left(161f_n - 37795f_{n+1} - 412860f_{n+2} - 535040f_{n+3} - 145805f_{n+4} - 2661f_{n+5}\right) \\
h^3y'''_{n+5} = \frac{5y_n}{2} - 11y_{n+1} + 18y_{n+2} - 13y_{n+3} + \frac{7y_{n+4}}{2} \\
+\frac{h^5}{60480}\left(170f_n + 5239f_{n+1} + 75876f_{n+2} + 108166f_{n+3} + 63434f_{n+4} + 4155f_{n+5}\right) \\
h^4y^{(4)}_{n+5} = y_n - 4y_{n+1} + 6y_{n+2} - 4y_{n+3} + y_{n+4} \\
+\frac{h^5}{60480}\left(853f_n - 2879f_{n+1} + 44902f_{n+2} + 35318f_{n+3} + 84149f_{n+4} + 19097f_{n+5}\right)
\end{cases}
\tag{2.5}
$$

Evaluating $Y^{(\mu)}(x)$, $\mu = 1(1)4$ of (2.4) at the point x_i, $i = 0(1)4$ and after some algebraic simplification, the following discrete 5-step LMM are obtained as the additional method;

$$
\begin{cases}
hy'_0 & = -\frac{25y_0}{12} + 4y_1 - 3y_2 + \frac{4y_3}{3} - \frac{y_4}{4} + \frac{h^5}{10080}\left(3f_0 + 749f_1 + 1194f_2 + 54f_3 + 19f_4 - 3f_5\right) \\
hy'_1 & = -\frac{y_0}{4} - \frac{5y_1}{6} + \frac{3y_2}{2} - \frac{y_3}{2} + \frac{y_4}{12} + \frac{h^5}{10080}\left(3f_0 - 120f_1 - 361f_2 - 21f_3 - 6f_4 + f_5\right) \\
hy'_2 & = \frac{y_0}{12} - \frac{2y_1}{3} + \frac{2y_2}{3} - \frac{y_4}{12} - \frac{h^5}{10080}\left(f_0 - 44f_1 - 250f_2 - 44f_3 + f_4\right) \\
hy'_3 & = -\frac{y_0}{12} + \frac{y_1}{2} - \frac{3y_2}{2} + \frac{5y_3}{6} + \frac{y_4}{4} - \frac{h^5}{10080}\left(36f_1 + 341f_2 + 135f_3 - 9f_4 + f_5\right) \\
hy'_4 & = \frac{y_0}{4} - \frac{4y_1}{3} + 3y_2 - 4y_3 + \frac{25y_4}{12} + \frac{h^5}{10080}\left(f_0 + 99f_1 + 1134f_2 + 794f_3 - 15f_4 + 3f_5\right) \\
h^2y''_0 & = \frac{35y_0}{12} - \frac{26y_1}{3} + \frac{19y_2}{2} - \frac{14y_3}{3} + \frac{11y_4}{12} - \frac{h^5}{302400}\left(2661f_0 + 109055f_1 + 130790f_2 + 8610f_3 + 1045f_4 - 161f_5\right) \\
h^2y''_1 & = \frac{11y_0}{12} - \frac{5y_1}{3} + \frac{y_2}{2} + \frac{y_3}{3} - \frac{y_4}{12} - \frac{h^5}{302400}\left(161f_0 - 9855f_1 - 15580f_2 + 520f_3 - 525f_4 + 79f_5\right) \\
h^2y''_2 & = -\frac{y_0}{12} + \frac{4y_1}{3} - \frac{5y_2}{2} + \frac{4y_3}{3} - \frac{y_4}{12} + \frac{h^5}{302400}\left(79f_0 - 1685f_1 - 510f_2 + 2450f_3 - 385f_4 + 51f_5\right) \\
h^2y''_3 & = -\frac{y_0}{12} + \frac{y_1}{3} + \frac{y_2}{2} - \frac{5y_3}{3} + \frac{11y_4}{12} - \frac{h^5}{302400}\left(51f_0 + 665f_1 + 14000f_2 + 11040f_3 - 635f_4 + 79f_5\right) \\
h^2y''_4 & = \frac{11y_0}{12} - \frac{14y_1}{3} + \frac{19y_2}{2} - \frac{26y_3}{3} + \frac{35y_4}{12} + \frac{h^5}{302400}\left(79f_0 + 11025f_1 + 127570f_2 + 111470f_3 + 1695f_4 + 161f_5\right) \\
h^3y'''_0 & = -\frac{5y_0}{2} + 9y_1 - 12y_2 + 7y_3 - \frac{3y_4}{2} + \frac{h^5}{60480}\left(4155f_0 + 57134f_1 + 38866f_2 + 6576f_3 - 1061f_4 + 170f_5\right) \\
h^3y'''_1 & = -\frac{3y_0}{2} + 5y_1 - 6y_2 + 3y_3 - \frac{y_4}{2} - \frac{h^5}{60480}\left(170f_0 - 1395f_1 - 13004f_2 - 686f_3 - 246f_4 + 41f_5\right) \\
h^3y'''_2 & = -\frac{y_0}{2} + y_1 - y_3 + \frac{y_4}{2} + \frac{h^5}{60480}\left(41f_0 - 1676f_1 - 11850f_2 - 1676f_3 + 41f_4\right) \\
h^3y'''_3 & = \frac{y_0}{2} - 3y_1 + 6y_2 - 5y_3 + \frac{3y_4}{2} + \frac{h^5}{60480}\left(1301f_1 + 12184f_2 + 2010f_3 - 416f_4 + 41f_5\right) \\
h^3y'''_4 & = \frac{3y_0}{2} - 7y_1 + 12y_2 - 9y_3 + \frac{5y_4}{2} - \frac{h^5}{60480}\left(41f_0 - 4026f_1 - 42266f_2 - 54584f_3 - 5175f_4 + 170f_5\right) \\
h^4y^{(iv)}_0 & = y_0 - 4y_1 + 6y_2 - 4y_3 + y_4 - \frac{h^5}{60480}\left(19097f_0 + 81629f_1 + 7598f_2 + 17182f_3 - 5399f_4 + 853f_5\right) \\
h^4y^{(iv)}_1 & = y_0 - 4y_1 + 6y_2 - 4y_3 + y_4 + \frac{h^5}{60480}\left(853f_0 - 21695f_1 - 41114f_2 + 3062f_3 - 1867f_4 + 281f_5\right) \\
h^4y^{(iv)}_2 & = y_0 - 4y_1 + 6y_2 - 4y_3 + y_4 - \frac{h^5}{60480}\left(281f_0 - 5059f_1 - 1810f_2 + 7774f_3 - 1367f_4 + 181f_5\right) \\
h^4y^{(iv)}_3 & = y_0 - 4y_1 + 6y_2 - 4y_3 + y_4 + \frac{h^5}{60480}\left(181f_0 + 1153f_1 + 35494f_2 + 25910f_3 - 2539f_4 + 281f_5\right) \\
h^4y^{(iv)}_4 & = y_0 - 4y_1 + 6y_2 - 4y_3 + y_4 - \frac{h^5}{60480}\left(281f_0 - 4387f_1 - 24658f_2 - 68834f_3 - 24215f_4 + 853f_5\right)
\end{cases}
\tag{2.6}
$$

Remark 2.1. All the formulae in (2.5) - (2.6) for $n = 0(5)N - 5$ considered together form the BVM which gives a system of 5N equations, with additional five boundary conditions leads to a system of $5N + 5$ equations in $5N + 5$ unknowns $\{y_j\}$ for $j = 0(1)N$.

3 Analysis of the Method

Method in (2.4) is associated with LMM of the form:

$$y(x) = \sum_{i=0}^{4} \alpha_i y_{n+i} + h^5 \sum_{i=0}^{5} \beta_i f_{n+i} \tag{3.1}$$

Thus, the linear differential operator $\mathcal{L}[y(x);\ h]$ is defined by

$$\mathcal{L}[y(x);\ h] = h^l y^{(l)}(x + ih) - \left(\sum_{i=0}^{4} \alpha_i y(x + ih) + h^5 \sum_{i=0}^{5} \beta_i y^{(v)}(x + ih)\right) \quad l = 0(1)4 \tag{3.2}$$

Expanding (3.2) in Taylor series, we obtain

$$\mathcal{L}[y(x);\ h] = C_0 y(x) + C_1 h y'(x) + C_2 h^2 y''(x) + \cdots + C_p h^q y^{(p)}(x) + O(h^{(p+1)}) \tag{3.3}$$

The LMM (3.1) is of order p if $C_0 = C_1 = C_2 = \cdots = C_{p+4} = 0$, and $C_{p+5} \neq 0$ in which

$$\mathcal{L}[y(x);\ h] = C_{p+5} h^{p+5} y^{(p+5)}(x) + O(h^{(p+6)}) \tag{3.4}$$

In this case, C_{p+5} is the error constant, see [16].

$$C_{p+5} = (\tau_{n+1}, \tau_{n+2}, \tau_{n+3}, \tau_{n+4}, \tau_{n+5})^T = \left(\frac{41}{221760}, -\frac{79}{362880}, -\frac{2321}{907200}, \frac{13}{1152}, -\frac{1}{6048}\right)^T$$

$$C_{p+5} = (\tau'_{n+1}, \tau'_{n+2}, \tau'_{n+3}, \tau'_{n+4}, \tau'_{n+5})^T = \left(-\frac{19}{266112}, \frac{1}{71280}, \frac{37}{1330560}, -\frac{5}{44352}, -\frac{91}{570240}\right)^T$$

$$C_{p+5} = (\tau''_{n+1}, \tau''_{n+2}, \tau''_{n+3}, \tau''_{n+4}, \tau''_{n+5})^T = \left(\frac{37}{226800}, -\frac{17}{201600}, \frac{89}{907200}, -\frac{571}{1814400}, -\frac{1}{3780}\right)^T$$

$$C_{p+5} = (\tau'''_{n+1}, \tau'''_{n+2}, \tau'''_{n+3}, \tau'''_{n+4}, \tau'''_{n+5})^T = \left(\frac{227}{453600}, -\frac{43}{453600}, -\frac{23}{129600}, \frac{229}{907200}, -\frac{337}{113400}\right)^T$$

$$C_{p+5} = (\tau^{(iv)}_{n+1}, \tau^{(iv)}_{n+2}, \tau^{(iv)}_{n+3}, \tau^{(iv)}_{n+4}, \tau^{(iv)}_{n+5})^T = \left(-\frac{361}{120960}, \frac{181}{120960}, -\frac{67}{40320}, \frac{341}{120960}, -\frac{277}{24192}\right)^T$$

We thus establish the convergence of the BVMs in the following theorem:

Theorem 3.1. *(Jator and Manathunga [17]) Let Y, $\overline{Y}$, and E be as defined above. Let $\overline{Y}$ be an approximation of the solution vector Y for the system formed by combining the methods* (2.5) *and* (2.6) *and $e_i = |y(x_i) - y_i|$, $he'_i = |hy'(x_i) - hy'_i|$, $h^2 e''_i = |h^2 y''(x_i) - h^2 y''_i|$, $h^3 e'''_i = |h^3 y'''(x_i) - h^3 y'''_i|$ and $h^4 e_i^{(4)} = |h^4 y^{(4)}(x_i) - h^4 y_i^{(4)}|$ be as defined above for $i = 1, \ldots, N$ where the exact solution $Y(x) \in C^n[a, b]$. Define $||E||_\infty = ||Y - \overline{Y}||_\infty$, then the BVMs is a sixth-order convergent method. That is $||E||_\infty = O(h^6)$.*

Proof. We write (2.5) together with (2.6) in the exact form

$$AY - h^5 BF(Y) + C + L(h) = 0 \tag{3.5}$$

where A is an $5N \times 5N$ matrix defined by

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} \\ A_{21} & A_{22} & A_{23} & A_{24} & A_{25} \\ A_{31} & A_{32} & A_{33} & A_{34} & A_{35} \\ A_{41} & A_{42} & A_{43} & A_{44} & A_{45} \\ A_{51} & A_{52} & A_{53} & A_{54} & A_{55} \end{bmatrix}$$

with A_{ij} an $N \times N$ matrices given as

$$A_{11} = \begin{bmatrix} -4 & 3 & -4/3 & 1/4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 26/3 & -19/2 & 14/3 & -11/12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ -9 & 12 & 7 & 3/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 4 & -6 & 4 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 5 & -10 & 10 & -5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 5 & -10 & 10 & -5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 5 & -10 & 10 & -5 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 5 & -10 & 10 & -5 & 0 & 0 & 0 & 0 & 0 & \cdots & \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 5 & -10 & 10 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 5 & -10 & 10 & -5 \end{bmatrix}$$

$$A_{21} = \begin{bmatrix} 5/6 & 3/2 & 1/2 & -1/12 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 2/3 & 0 & -2/3 & -1/12 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ -1/2 & 3/2 & -5/6 & -1/4 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 4/3 & -3 & 4 & 25/12 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 61/6 & -39/2 & 107/6 & 77/12 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 61/6 & -39/2 & 107/6 & 77/12 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 61/6 & -39/2 & 107/6 & 77/12 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 61/6 & -39/2 & 107/6 & 77/12 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 61/6 & -39/2 & 107/6 & 77/12 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 61/6 & -39/2 & 107/6 & 77/12 \end{bmatrix}$$

$$A_{31} = \begin{bmatrix} 5/3 & -1/2 & -1/3 & 1/12 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ -4/3 & 5/2 & -4/3 & 1/12 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ -1/3 & -1/2 & -5/3 & -11/12 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 14/3 & -19/2 & 26/3 & -35/12 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 41/3 & -49/2 & 59/3 & 71/12 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 41/3 & -49/2 & 59/3 & 71/12 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 41/3 & -49/2 & 59/3 & 71/12 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 41/3 & -49/2 & 59/3 & 71/12 & 0 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 41/3 & -49/2 & 59/3 & 71/12 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 41/3 & -49/2 & 59/3 & 71/12 \end{bmatrix}$$

$$A_{41} = \begin{bmatrix} -5 & 6 & -3 & 1/2 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 0 & 1 & -1/2 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ -3 & -6 & -5 & -3/2 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 7 & -12 & 9 & -5/2 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 11 & -18 & 13 & -7/2 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 11 & -18 & 13 & -7/2 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 11 & -18 & 13 & -7/2 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 11 & -18 & 13 & -7/2 & 0 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 11 & -18 & 13 & -7/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 11 & -18 & 13 & -7/2 \end{bmatrix}$$

$$A_{51} = \begin{bmatrix} 4 & -6 & 4 & -1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 4 & -6 & 4 & -1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 4 & -6 & 4 & -1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 4 & -6 & 4 & -1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 4 & -6 & 4 & -1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 4 & -6 & 4 & -1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 4 & -6 & 4 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 4 & -6 & 4 & -1 & 0 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 4 & -6 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 4 & -6 & 4 & -1 \end{bmatrix}$$

$A_{22} = A_{33} = A_{44} = A_{55} = I$, where I is an $N \times N$ identity matrices and $A_{ij} = 0$ for $i = 1, 2, 3, 4, 5$; $j = 2, 3, 4, 5$; $i \neq j$.

Similarly, B is a $5N \times 5N$ matrix defined by

$$B = \begin{bmatrix} B_{11} & B_{12} & B_{13} & B_{14} & B_{15} \\ B_{21} & B_{22} & B_{23} & B_{24} & B_{25} \\ B_{31} & B_{32} & B_{33} & B_{34} & B_{35} \\ B_{41} & B_{42} & B_{43} & B_{44} & B_{45} \\ B_{51} & B_{52} & B_{53} & B_{54} & B_{55} \end{bmatrix}$$

with B_{ij} an $N \times N$ matrices given as

$$B_{11} = \begin{bmatrix} \frac{107}{1440} & \frac{199}{1680} & \frac{3}{560} & \frac{19}{10080} & \frac{-1}{3360} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \frac{-21811}{60480} & \frac{-13079}{30240} & \frac{-41}{1440} & \frac{-209}{60480} & \frac{23}{43200} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \frac{4081}{4320} & \frac{19433}{30240} & \frac{137}{1260} & \frac{-1061}{60480} & \frac{17}{6048} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \frac{-81629}{60480} & \frac{-3799}{30240} & \frac{-8591}{30240} & \frac{-1061}{60480} & \frac{-853}{60480} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \frac{1}{24} & \frac{11}{24} & \frac{11}{24} & \frac{1}{24} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{24} & \frac{11}{24} & \frac{11}{24} & \frac{1}{24} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{24} & \frac{11}{24} & \frac{11}{24} & \frac{1}{24} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \frac{1}{24} & \frac{11}{24} & \frac{11}{24} & \frac{1}{24} & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \frac{1}{24} & \frac{11}{24} & \frac{11}{24} & \frac{1}{24} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \frac{1}{24} & \frac{11}{24} & \frac{11}{24} & \frac{1}{24} & 0 \end{bmatrix}$$

$$B_{21} = \begin{bmatrix} \frac{-1}{84} & \frac{-361}{10080} & \frac{-1}{480} & \frac{-1}{1680} & \frac{1}{10080} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \frac{11}{2520} & \frac{25}{1008} & \frac{11}{2520} & \frac{-1}{10080} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \frac{-1}{280} & \frac{-341}{10080} & \frac{-3}{224} & \frac{1}{1120} & \frac{-1}{10080} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \frac{11}{1120} & \frac{9}{80} & \frac{397}{5040} & \frac{-1}{672} & \frac{1}{3360} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \frac{149}{1680} & \frac{9679}{10080} & \frac{10819}{10080} & \frac{29}{180} & \frac{1}{3360} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \frac{149}{1680} & \frac{9679}{10080} & \frac{10819}{10080} & \frac{29}{180} & \frac{1}{3360} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \frac{149}{1680} & \frac{9679}{10080} & \frac{10819}{10080} & \frac{29}{180} & \frac{1}{3360} & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \frac{149}{1680} & \frac{9679}{10080} & \frac{10819}{10080} & \frac{29}{180} & \frac{1}{3360} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \frac{149}{1680} & \frac{9679}{10080} & \frac{10819}{10080} & \frac{29}{180} & \frac{1}{3360} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \frac{149}{1680} & \frac{9679}{10080} & \frac{10819}{10080} & \frac{29}{180} & \frac{1}{3360} \end{bmatrix}$$

$$B_{31}=\begin{bmatrix}
\frac{73}{2240} & \frac{779}{15120} & \frac{-13}{7560} & \frac{1}{576} & \frac{-79}{302400} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0\\
\frac{-337}{60480} & \frac{-17}{10080} & \frac{7}{864} & \frac{-11}{8640} & \frac{17}{100800} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0\\
\frac{-19}{8640} & \frac{-5}{108} & \frac{-23}{630} & \frac{127}{60480} & \frac{-79}{100800} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0\\
\frac{7}{192} & \frac{12757}{30240} & \frac{11147}{30240} & \frac{113}{20160} & \frac{23}{43200} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0\\
\frac{7559}{60480} & \frac{983}{720} & \frac{1672}{945} & \frac{29161}{60480} & \frac{887}{100800} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0\\
0 & \frac{7559}{60480} & \frac{983}{720} & \frac{1672}{945} & \frac{29161}{60480} & \frac{887}{100800} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0\\
0 & 0 & \frac{7559}{60480} & \frac{983}{720} & \frac{1672}{945} & \frac{29161}{60480} & \frac{887}{100800} & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0\\
0 & 0 & 0 & \frac{7559}{60480} & \frac{983}{720} & \frac{1672}{945} & \frac{29161}{60480} & \frac{887}{100800} & 0 & 0 & 0 & 0 & 0 & \cdots & 0\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \frac{7559}{60480} & \frac{983}{720} & \frac{1672}{945} & \frac{29161}{60480} & \frac{887}{100800} & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \frac{7559}{60480} & \frac{983}{720} & \frac{1672}{945} & \frac{29161}{60480} & \frac{887}{100800}
\end{bmatrix}$$

$$B_{41}=\begin{bmatrix}
\frac{31}{1344} & \frac{3251}{15120} & \frac{49}{4320} & \frac{41}{10080} & \frac{-41}{60480} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0\\
\frac{-419}{15120} & \frac{-395}{2016} & \frac{-419}{15120} & \frac{41}{60480} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0\\
\frac{1301}{60480} & \frac{1523}{7560} & \frac{67}{2016} & \frac{13}{1890} & \frac{41}{60480} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0\\
\frac{671}{10080} & \frac{3019}{4320} & \frac{6823}{7560} & \frac{115}{1344} & \frac{-17}{60480} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0\\
\frac{5239}{60480} & \frac{6323}{5040} & \frac{54083}{30240} & \frac{4531}{4320} & \frac{277}{4032} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0\\
0 & \frac{5239}{60480} & \frac{6323}{5040} & \frac{54083}{30240} & \frac{4531}{4320} & \frac{277}{4032} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0\\
0 & 0 & \frac{5239}{60480} & \frac{6323}{5040} & \frac{54083}{30240} & \frac{4531}{4320} & \frac{277}{4032} & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0\\
0 & 0 & 0 & \frac{5239}{60480} & \frac{6323}{5040} & \frac{54083}{30240} & \frac{4531}{4320} & \frac{277}{4032} & 0 & 0 & 0 & 0 & 0 & \cdots & 0\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \frac{5239}{60480} & \frac{6323}{5040} & \frac{54083}{30240} & \frac{4531}{4320} & \frac{277}{4032} & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \frac{5239}{60480} & \frac{6323}{5040} & \frac{54083}{30240} & \frac{4531}{4320} & \frac{277}{4032}
\end{bmatrix}$$

$$B_{51}=\begin{bmatrix}
\frac{-4339}{12096} & \frac{-20557}{30240} & \frac{1531}{30240} & \frac{-1867}{60480} & \frac{281}{60480} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0\\
\frac{5059}{60480} & \frac{181}{6048} & \frac{-3887}{30240} & \frac{1367}{60480} & \frac{-181}{60480} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0\\
\frac{1153}{60480} & \frac{17747}{30240} & \frac{2591}{6048} & \frac{-2539}{60480} & \frac{281}{60480} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0\\
\frac{4387}{60480} & \frac{12329}{30240} & \frac{34417}{30240} & \frac{4843}{12096} & \frac{-853}{60480} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0\\
\frac{-2879}{60480} & \frac{22481}{30240} & \frac{17659}{30240} & \frac{84149}{60480} & \frac{19097}{60480} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0\\
0 & \frac{-2879}{60480} & \frac{22481}{30240} & \frac{17659}{30240} & \frac{84149}{60480} & \frac{19097}{60480} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0\\
0 & 0 & \frac{-2879}{60480} & \frac{22481}{30240} & \frac{17659}{30240} & \frac{84149}{60480} & \frac{19097}{60480} & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0\\
0 & 0 & 0 & \frac{-2879}{60480} & \frac{22481}{30240} & \frac{17659}{30240} & \frac{84149}{60480} & \frac{19097}{60480} & 0 & 0 & 0 & 0 & 0 & \cdots & 0\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \frac{-2879}{60480} & \frac{22481}{30240} & \frac{17659}{30240} & \frac{84149}{60480} & \frac{19097}{60480} & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \frac{-2879}{60480} & \frac{22481}{30240} & \frac{17659}{30240} & \frac{84149}{60480} & \frac{19097}{60480}
\end{bmatrix}$$

where B_{ij} is an $N \times N$ zero matrix for $i = 1,2,3,4,5$; $j = 2,3,4,5$.

$$C = \Big(-hy_0', -\frac{25y_0}{12}, h^5\frac{f_0}{3360}, -h^2y_0'', \frac{35y_0}{12}, h^5\frac{-887f_0}{100800}, -h^3y_0''', -\frac{5y_0}{2}, h^5\frac{277f_0}{4032}, -h^4y_0^{(4)}, y_0, h^5\frac{-19097f_0}{60480},$$
$$y_0, 0, \ldots, 0, -\frac{y_0}{4}, h^5\frac{f_0}{3360}, \frac{y_0}{12}, -\frac{y_0}{12}, \frac{y_0}{4}, h^5\frac{f_0}{10080}, \frac{25y_0}{12}, h^5\frac{-f_0}{3360}, 0, \ldots, 0, \frac{11y_0}{12}, h^5\frac{79f_0}{302400}, -\frac{y_0}{12},$$
$$-h^5\frac{17f_0}{100800}, h^5\frac{79f_0}{302400}, \frac{35y_0}{12}, h^5\frac{-23f_0}{43200}, 0, \ldots, 0, -\frac{3y_0}{2}, -h^5\frac{17f_0}{6048}, -\frac{y_0}{2}, h^5\frac{41f_0}{60480}, \frac{y_0}{2}, \frac{3y_0}{2}, -h^5\frac{41f_0}{60480},$$
$$\frac{5y_0}{2}, h^5\frac{17f_0}{6048}, 0, \ldots, 0, y_0, h^5\frac{853f_0}{60480}, y_0, h^5\frac{-281f_0}{60480}, y_0, h^5\frac{181f_0}{60480}, y_0, h^5\frac{-281f_0}{60480}, y_0, h^5\frac{853f_0}{60480}, 0, \ldots, 0\Big)^T$$

where $L(h)$ is the truncation error of the formulas (2.5) and (2.6). defined as

$$L(h) = (\tau_1 \ldots, \tau_N, h\tau_1' \ldots, h\tau_N', h^2\tau_1'' \ldots, h^2\tau_N'', h^3\tau_1''' \ldots, h^3\tau_N''', h^4\tau_1^{(iv)} \ldots, h^4\tau_N^{(iv)})^T$$

and

$$Y = (y(x_1), \ldots, y(x_N), hy'(x_1), \ldots, hy'(x_N), h^2y''(x_1), \ldots, h^2y''(x_N), h^3y'''(x_1), h^3y'''(x_N), h^4y^{(iv)}(x_1), \ldots, h^4y^{(iv)}(x_N))^T$$

$$F(Y) = (f_1, \ldots, f_N, hf_1', \ldots, hf_N', h^2f_1'', \ldots, h^2f_N'', h^3f_1''', \ldots, h^3f_N''', h^4f_1^{(iv)}, \ldots, h^4f_N^{(iv)})^T$$

The approximate form of the system is given as

$$A\overline{Y} - h^5BF(\overline{Y}) + C = 0 \tag{3.6}$$

where $\bar{Y}$ is the approximate solution of the vector Y Define the vector E as

$$E = \overline{Y} - Y = (e_1, \ldots, e_N, he_1', \ldots, he_N', h^2e_1'', \ldots, h^2e_N'', h^3e_1''', \ldots, h^3e_N''', h^4e_1^{(iv)}, \ldots, h^4e_N^{(iv)})^T$$

where

$$\overline{Y} = (y_1, \ldots, u_N, hy_1', \ldots, hy_N', h^2y_1'', \ldots, h^2y_N'', h^3y_1''', \ldots h^3y_N''', h^4y_1^{(iv)}, \ldots h^4y_N^{(iv)})^T$$

Subtracting (3.5) from (3.6) and applying the mean value theorem, we obtain the error system as

$$(A - BJ_f)E = L(h) \tag{3.7}$$

where J_f is the Jacobian matrix

$$J_f = \begin{pmatrix} J_{11} & \cdots & J_{15} \\ \vdots & & \vdots \\ J_{51} & \cdots & J_{55} \end{pmatrix}$$

whose entries J_{ij} are $N \times N$ matrices given as $J_{11} = \begin{pmatrix} \frac{\partial f_1}{\partial y_1} & \cdots & \frac{\partial f_1}{\partial y_N} \\ \vdots & & \vdots \\ \frac{\partial f_N}{\partial y_1} & \cdots & \frac{\partial f_N}{\partial y_N} \end{pmatrix}$;

$$J_{1,j+1} = \begin{pmatrix} \frac{\partial f_1}{\partial y_1^{(j)}} & \cdots & \frac{\partial f_1}{\partial y_N^{(j)}} \\ \vdots & & \vdots \\ \frac{\partial f_N}{\partial y_1^{(j)}} & \cdots & \frac{\partial f_N}{\partial y_N^{(j)}} \end{pmatrix} \text{ for } j = 1, 2, 3, 4;$$

$$J_{i,j+1} = h^i \begin{pmatrix} \frac{\partial f_1^{(i-1)}}{\partial y_1^{(j)}} & \cdots & \frac{\partial f_1^{(i-1)}}{\partial y_N^{(j)}} \\ \vdots & & \vdots \\ \frac{\partial f_N^{(i-1)}}{\partial y_1^{(j)}} & \cdots & \frac{\partial f_N^{(i-1)}}{\partial y_N^{(j)}} \end{pmatrix}; \text{ for } j = 0, \ldots, 4. \text{ and for } i = 2, \ldots, 5.$$

Let $M = -BJ_f$ be a matrix of dimension $5N$ so that (3.7) becomes

$$(A + M)E = L(h) \tag{3.8}$$

and for sufficiently small h, $A + M$ is a monotone matrix and thus nonsingular (see[18]). Hence

$$\begin{array}{ll} (A+M)^{-1} & = D = (d_{ij}) >\geq 0 \\ \sum_{j=1}^{5M} d_{ij} & = O(h^{-5}), \\ E & = DL(h), \\ ||E|| & = ||DL(h)|| = O(h^{-5})O(h^{11}) \\ & = O(h^6) \end{array} \tag{3.9}$$

which shows that the method is 6th order convergent, that is, with a global error of order $O(h^6)$. □

3.1 Computational aspects

Implementation

The method (2.5) and (2.6) can be expressed in block form as

$$A_0V_\mu = A_1V_{\mu-1} + h^5B_1F_{\mu-1} + h^5B_0F_\mu, \quad \mu = 1, \ldots, \Gamma, \quad n = 0, 5, \ldots, N-5 \tag{3.10}$$

where

$$\begin{array}{l} V_\mu = (y_1, \ldots, y_{n+5}, hy_1', \ldots, hy_{n+5}', h^2y_1'', \ldots, h^2y_{n+5}'', h^3y_1''', \ldots, h^3y_{n+5}''', h^4y_1^{(iv)}, \ldots, h^4y_{n+5}^{(iv)})^T \\ V_{\mu-1} = (y_0, \ldots, y_{n-4}, hy_0', \ldots, hy_{n-4}', h^2y_0'', \ldots, h^2y_{n-4}'', h^3y_0''', \ldots, h^3y_{n-4}''', h^4y_0^{(iv)}, \ldots, h^4y_{n-4}^{(iv)})^T \\ F_{\mu-1} = (f_0, \ldots, f_{n-4}, hf_0', \ldots, hf_{n-4}', h^2f_0'', \ldots, h^2f_{n-4}'', h^3f_0''', \ldots, h^3f_{n-4}''', h^4f_0^{(iv)}, \ldots, h^4f_{n-4}^{(iv)})^T \\ F_\mu = (f_1, \ldots, f_{n+5}, hf_1', \ldots, hf_{n+5}', h^2f_1'', \ldots, h^2f_{n+5}'', h^3f_1''', \ldots, h^3f_{n+5}''', h^4f_1^{(iv)}, \ldots, h^4f_{n+5}^{(iv)})^T \end{array}$$

The positive integer $\Gamma = \frac{N}{5}$ is the number of blocks.
The Boundary Value Methods has been implemented using the system *Mathematica*, enhanced by

the feature ***NSolve[]*** for linear problems while nonlinear problems were solved by Newton's method enhanced by the feature ***FindRoot[]***, as summarized in the algorithm below. We begin by noting that the solution of the problem (1.1) is sought in the subinterval $\pi_N = \{a = x_0 < x_1 < \ldots < x_N = b\}$, where $h = \frac{b-a}{N}$ is a constant step-size of the partition of π_N, N is a positive integer and n the grid index.

We emphasize the methods (2.5) together with (2.6) lead to a single matrix of finite difference equations, which is solved to provide all the solutions of (1.1) on the entire interval $[a, b]$.
Step 1: Use the block of (3.10) for $\mu = 1$, $n = 0$ to obtain V_1 on the rectangle $[y_0, y_5] \times [a, b]$, similarly, for $\mu = 2$, $n = 5$ so that V_2 is obtained on the rectangle $[y_5, y_{10}] \times [a, b]$, and on the rectangle $[y_{10}, y_{15}] \times [a, b], \ldots, [y_{N-5}, y_N] \times [a, b]$ for $\mu = 3, \ldots \Gamma$, $n = 10, 15, \ldots, N-5$, we thus obtain $\mathrm{V}_3, \ldots, \mathrm{V}_\Gamma$.
Step 2: Solve the unified block given by the system $\mathrm{V}_1 \bigcup \mathrm{V}_2 \bigcup \ldots \bigcup \mathrm{V}_\Gamma$, obtained in step 1.
Step 3:The solution of (1.1) is approximated by the solutions in step 2 as $y = [y(x_1), y(x_2), \ldots, y(x_n)]^T$, $n = 1, 2, \ldots, N$.

Algorithm.

Data: a, b (integration interval), N (number of steps)
Input: Impute boundary conditions from the problem.
Input: Adjust conditions so as to determine other unknown conditions
Output: sol, discrete approximate solution of BVMs (2.5)-(2.6)

1 For $n = 0, 5, \ldots, N-5$ in (2.5), Combine (2.5) and (2.6), For $\mu = 1, 2, \ldots, \Gamma$;
2 Generate $\mathrm{V}_1, \mathrm{V}_2, \ldots, \mathrm{V}_\Gamma$, for each n and μ;
3 Set System $\mathrm{V}_1 \bigcup \mathrm{V}_2 \bigcup \ldots \bigcup \mathrm{V}_\Gamma$;
4 Solve NSolve[System, data] if linear, otherwise go to 5;
5 Solve FindRoot[System, data],;
6 Let $sol = y(x_n)$. for $n = 1, 2, \ldots, N$;
7 **if** $n = N$ **then**
8 | go to 11
9 **else**
10 | go to 4;
11 **end**
12 End

4 Numerical Examples

In this section, some numerical examples are used to illustrate the accuracy and efficiency of the method. It is found that the maximum absolute error (Err) of the approximate solution on the partitions

$$\pi_N = \{a = x_0 < x_1 < x_2 < \cdots < x_N = b\}$$

is given as $Err = Max||y(x_n) - y_n||$, with constant step-size $h = (b-a)/N$. N is the number of partitions or subintervals of $[x_n, x_{n+5}]$. This numerical experiments was done using codes written in Wolfram® Mathematica 8.0.

Example 1. Consider the nonlinear fifth order boundary value problem

$$y^{(5)}(x) + y^{(4)}(x) + e^{-2x}y^2 = 2e^x + 1, \quad 0 \le x \le 1$$
$$y(0) = 1, \quad y(1) = e, \quad y'(0) = 1, \quad y'(1) = e, \quad y''(0) = 1$$

The exact solution is given by $y(x) = e^x$.
In this example, we have compared the BVM with Reproducing Kernel Space Method (RKSM) in [19], the Embedded Perturbed Chebyshev Integral Collocation Method (EPCICM) and Embedded Perturbed Bernstein Integral Collocation Method (EPBICM) both in [20]. It can be clearly seen that the BVM is superior in terms of the maximum errors obtained for this problem.

Table 1: Error for Problem 1, $h = 0.1$

Methods	Maximum Absolute error
BVM	1.306×10^{-13}
RKSM	1.0178×10^{-07}
EPCICM	1.121×10^{-05}
EPBICM	3.049×10^{-05}

Example 2. Consider the nonlinear fifth order boundary value problem

$$y^{(5)} - e^{-x}y^2 = 0, \quad 0 \leq x \leq 1,$$
$$y(0) = y'(0) = y''(0) = 1, \quad y(1) = y'(1) = e.$$

with theoretical solution $y(x) = e^x$.

Table 2 shows the absolute errors obtained by using the Boundary Value Method (BVM), Variational Iteration Method (VIM) in [21], Homotopy Perturbation Method (HPM) in [22], Variation of Parameters (VOP) in [23] and the Adomians Decomposition Method (ADM) in [24]. Is evident that The BVM performs favourably well as compared to the other methods.

Table 2: Error of methods for Example 2, $h = 0.1$

x	BVM	VIM	HPM	VOP	ADM
0.1	2.66×10^{-15}	1.0×10^{-09}	1.0×10^{-09}	1.3×10^{-12}	1.0×10^{-09}
0.2	9.99×10^{-15}	2.0×10^{-09}	2.0×10^{-09}	1.0×10^{-11}	2.0×10^{-09}
0.3	2.39×10^{-14}	1.0×10^{-08}	1.0×10^{-08}	3.2×10^{-11}	1.0×10^{-08}
0.4	4.19×10^{-14}	2.0×10^{-08}	2.0×10^{-08}	7.0×10^{-11}	2.0×10^{-08}
0.5	5.79×10^{-14}	3.1×10^{-08}	3.1×10^{-08}	1.2×10^{-10}	3.1×10^{-08}
0.6	6.51×10^{-14}	3.7×10^{-08}	3.7×10^{-08}	1.9×10^{-10}	3.7×10^{-08}
0.7	5.86×10^{-14}	4.1×10^{-08}	4.1×10^{-08}	2.8×10^{-10}	4.1×10^{-08}
0.8	3.86×10^{-14}	3.1×10^{-08}	3.1×10^{-08}	3.7×10^{-10}	3.1×10^{-08}
0.9	1.24×10^{-14}	1.4×10^{-08}	1.4×10^{-08}	4.7×10^{-10}	1.4×10^{-08}
1.0	0.00	0.00000	0.00000	5.6×10^{-10}	0.00000

Example 3. Consider the following fifth-order boundary value problem [5]

$$\begin{cases} y^{(5)} + 24e^{-5y} = \frac{48}{(1+x)^5} \\ y(0) = 0, \;\; y(1) = \ln 2, \\ y'(0) = 1, \;\; y'(1) = 0.5, \\ y''(0) = -1 \end{cases}$$

Its exact solution is $y(x) = \ln(1 + x)$.
Table 3 presents the maximum absolute error obtained using the BVM, methods in [19] and [25]. The errors obtained show that the BVM is superior.

Table 3: Error of methods for Problem 3

Method	Maximum Error
BVM ($h = 0.05$)	1.1×10^{-10}
RKSM (n=10)	9.2×10^{-06}
Method in [25]	5.0×10^{-05}

Example 4. Consider the following fifth-order boundary value problem [26]

$$\begin{cases} y^{(5)}(x) - 4y'(x) = 0 \\ y(0) = 0, \quad y'(0) = 1, \\ y''(0) = 2, \quad y(1) = \exp(1)\sin(1), \\ y'(1) = \exp(1)(\sin(1) + \cos(1)) \end{cases}$$

Its exact solution is $y(x) = \exp(x)\sin(x)$. In Table 4, at different values of N as above, it shows the

Table 4: Maximum Absolute Error for Problem 4

N	BVM	Method in [26]
16	4.90719×10^{-14}	6.2513351×10^{-4}
32	3.83693×10^{-13}	1.6224384×10^{-4}
64	1.37772×10^{-11}	4.4584274×10^{-5}
128	2.77446×10^{-10}	1.1444092×10^{-5}

maximum absolute errors obtained, in comparison with the BVM and the method of Pandey [26]. This shows the superiority of the BVM in terms of the errors obtained.

Example 5. Consider the following fifth-order boundary value problem [26]

$$\begin{cases} y^{(5)}(x) - \frac{(y'(x))^2}{(5+x)^3} = \frac{23}{(5+x)^5}, \quad 0 < x < 1, \\ y(0) = \ln(5), \quad y'(0) = \frac{1}{5}, \\ y''(0) = -\frac{1}{25}, \quad y(1) = \ln(6)\sin(1), \\ y'(1) = \frac{1}{6} \end{cases}$$

Its exact solution is $y(x) = \ln(x + 5)$. Similarly, in Table 5, at different values of N, shows the

Table 5: Maximum Absolute Error for Problem 5

N	BVM	Method in [26]
16	3.37508×10^{-14}	1.9752979×10^{-4}
32	5.58664×10^{-15}	4.3869019×10^{-5}
64	1.78328×10^{-17}	1.1444092×10^{-5}
128	6.60452×10^{-19}	2.5033951×10^{-6}

maximum errors obtained, in comparison with the BVM and the method of Pandey [26]. Again, this shows the superiority of the BVM in terms of the errors obtained.

Example 6. Consider the following fifth-order boundary value problem [5]

$$\begin{cases} y^{(5)}(x) = y(x) - 15\,\mathrm{e}^{x} - 10\,x\,\mathrm{e}^{x} \\ y(0) = 0, \;\; y(1) = 1, \\ y'(0) = 1, \;\; y'(1) = -e, \\ y''(0) = 0. \end{cases}$$

Its exact solution is $y(x) = x(1-x)e^x$. The BVM for small h was compared to the Nonsymmetric

Table 6: Absolute Error for Problem 6. $h = 0.001$

Methods	Maximum error
BVM	1.84×10^{-25}
GJPGM	6.35×10^{-17}
Sextic spline method	4.84×10^{-7}
Cubic B-spline method	1.14×10^{-5}

Generalized Jacobi PetrovGalerkin method GJPGM in [27], Sextic spline method in [14] and Cubic B-spline method in [18]. It is obvious that the BVM performed better in this problem.

5 Conclusion

We have derived continuous block finite difference methods which has be implemented as the boundary value method (BVM) to solve $y^{(v)} = f(x, y, y', y'', \ldots, y^{(iv)})$ subject to appropriate boundary conditions. This was carried out without first reducing the ODE to an equivalent first order system. The beauty of this method is that it is self starting as it does not require any starting value as do by other methods. The implementation is less time costly. The numerical experiments performed shows the efficiency, accuracy and advantages of the method over existing ones in the literature as regards the problems considered.

Competing Interests

Authors have declared that no competing interests exist.

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