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Population Dynamics in Optimally Controlled Economic Growth Models: Case of Linear Production Function

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Authors' contributions

This work was carried out in collaboration between all authors. Author SOB designed the study, performed the statistical analysis, wrote the protocol, wrote the first draft of the manuscript, performed the analyses and managed the literature searches. Authors FTO and GAO directed, supervised and managed the analyses of the study. All authors read and approved the final manuscript.

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Abstract

This paper discusses optimally controlled economic growth models with linear aggregate production function of capital and labour. It compares and contrasts real per capita income performance over time in situations where the labour (population) growth dynamics vary from purely exponential to strongly logistic. The study seeks to identify, by means of analytical and qualitative methods, as well as numerical simulations, the causal factors and parameters, particularly population related ones, which induce qualitative changes in the performance of real per capita income over time. Furthermore, the paper discusses the concept of maximum sustainable population growth for the models and the conditions for exiting the Malthusian trap. Consumption per (effective) labour is used as the control variable, and capital per (effective) labour, as state variable, whereas income per (effective) labour is considered the output variable. A time-discounted welfare functional is applied as the models' objective functional, maximized subject to a differential equation in the control and state variables. Each system is found to be controllable and observable. The models' simulation results are reasonably intuitive and realistic. The results also indicate that, consistently, real per capita income grows faster and generates greater time values, however marginal, as the population growth dynamics tends increasingly logistic.

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Keywords: Population dynamics; logistic growth; Malthusian growth; economic development; optimal control.

1 Introduction

Economic growth theory suggests that the aggregate production function of an economy, which generally determines the time-evolution of gross domestic product (GDP), and hence, real per capita income in an economy, is predicated on the level and quality of labour, technology, physical and human capital [1,2,3,4,5,6]. Recent studies [7,8,9,10] also lend credence to these assertions. Other salient factors such as natural resources, international trade, fiscal and monetary policy options, and even some more, may also count.

One such factor of interest among these other factors, controversial in economics literature though, is the issue of population, its size and growth dynamics, in relation to real per capita income over time. [11] states, inter alia, that, high population growth puts a lot of strain on economic performance, the presence of technology notwithstanding. However, optimists such as [12] believe that through technological innovations and advancement, the earth's capability of containing humans is boundless. In contrast to these extreme views, [13] states that population growth boosts technological growth, which in turn enhances economic well-being. Subsequently, as resources begin to run out, an economy is forced to invent its way out of the problem.

In spite of the seeming flaws in Malthus' prepositions, they sum up the economies of the mediaeval world, and up to his (Malthus') days. This was a time when the industrial revolution of England, preceded by its agricultural revolution, over half a century earlier, was already some couple of decades old. Though it initially begat a rich middle class, real per capita incomes of the working class saw marginal or no improvement [14]. England's second phase of industrial revolution, starting around the 1850s, resulted in a steady but sustained growth in its real per capita income, rising sharply before 1900. This rise in real per capita incomes resulted in improved healthcare, economic well-being, education and skill accumulation, which in turn, led to the advances in technology, as well as the capitalization of labour [14]. At the same time, the rates of population growth (when exponential) and capital depreciation, adversely impact on real per capita income over time. Equally, the population growth dynamics is an essential ingredient in the determination of the time-value of real per capita income.

The break out from the poverty (Malthusian) trap into sustained rise in real per capita incomes of the Western European, North American, and most developed economies of today, have followed similar tracks and traits. Contemporary examples abound: the Asian tigers like South Korea, Singapore, China and Malaysia. Initially as real per capita income rises, population rises, due mainly to access to improved healthcare, and hence a reduction in mortality, and a rise in life expectancy. As these economies surge forward into higher real per capita incomes, with its attendant expanded access to education, improved economic opportunities and exposure, population growth slows down.

The improved labour productivity, induced by the growing use of technology, physical and human capital, grows the economy, and hence, real per capita GDP. [15] cites the USA's example, in which its 2.1 percent annual real growth rate in labour productivity, over a century, converted to about 2.1 percent real annual growth in real per capita GDP, doubling living standards every 35 years.

Nonetheless, empirical contemporary economic realities of undeveloped and developing countries show the prevalence of high population growth rate, low levels of capital stock, technological progress and investment drive, as a result of low savings. Impliedly, to a large extent, these economies ostensibly wallow in a Malthusian trap, and find it extremely difficult to break away from this trap. Even though economic history of developed economies tells us that this problem is resolvable, there is hardly any workable deterministic models that seek to address this. Again, Malthus' and Boserup's positions innately suggest some element of population carrying capacity. But there is scarcely any in-depth mathematical analysis on the effects of population growth dynamics on economic performance.

Impliedly, the economic history of developed economies suggests that the problems posed by Malthus can be surmounted, using methods including, but not limited to:

a) The population (labour) becoming more efficient through increasing deployment of technology.
b) The population / labour partly assuming the form of capital stock (i.e., human capital) through the acquisition of more knowledge, training and skills.

However, there is the need in this regard, to go beyond the largely empirical studies in the current literature. Hence, formulate rigorous mathematical model(s) to capture, explain and predict the dependence of economic performance on population dynamics and its relation to the other system parameters, especially including technology and human capital.

In this paper, however, we deliberate on the impact of population dynamics in optimally controlled economic growth models, using a linear aggregate production function of capital and labour (and technology). Local system controllability and stability are discussed. The discussion also looks at the idea of maximum sustainable population growth, its effects on real per capita income performance. It also performs qualitative analyses on the models with respect to their dependence on the system parameters, particularly population related ones. It further implements numerical simulations on the models to confirm the suspected dependence or otherwise.

2 Theoretical Preliminaries

2.1 Linear control

Let $u \in R$, $x \in R$, and $y \in R$ respectively denote the control, state and output variables of a system, with representation $\{a(t), b(t), c(t), d(t)\}$. The control variable varies within a fixed and pre-specified control set [16, 17, 18, 19, 20, 21]. Then the state and output equations of the system are given by

$$\dot{x}(t) = g(x(t), u(t), p, t) = a(t)x(t) + b(t)u(t) \tag{2.1.1}$$

$$y(t) = G(x(t), u(t), t) = c(t)x(t) + d(t)u(t) \tag{2.1.2}$$

where p denotes the set of system parameters.

2.2 Optimal control problem and the Hamilton-Pontryagin equations

The optimal control problem primarily seeks to maximize an objective (welfare) functional,

$$W(x, u) = \int_{t_0}^{T_f} L(x(t), u(t), t)dt \quad \text{for } T_f \geq 0 \tag{2.2.1}$$

subject to $\dot{x}(t) = g(x(t), u(t), p, t)$, for $x(t_0) = x_0 \geq 0$ and $x(T_f) = x_{T_f} \geq 0$.

From the above, a Hamiltonian function, H, is constructed thus

$$H = H(x(t), u(t), \lambda(t), t) = L(x(t), u(t), t) + \lambda(t)g(x(t), u(t), p, t), \tag{2.2.2}$$

λ is the co-state function. The associated Hamilton-Pontryagin [22] equations are:

$$H_x = L_x + \lambda g_x = -\dot{\lambda} \tag{2.2.3}$$

$$H_\lambda = g = \dot{x} \tag{2.2.4}$$

$$H_u = L_u + \lambda g_u = 0 \tag{2.2.5}$$

for $x(t_0) = x_0 \geq 0$, $x(T_f) = x_{T_f} \geq 0$ or $\lambda(T_f) = Px(T_f)$, for some non-zero value P.

2.3 Stability, controllability and observability criteria

The scalar system provided in Equation (2.1.1) above is stable if and only if $a(t) < 0$, for all t within the interval $[t_0, T_f]$. For instance, [18] suggests that the system is controllable within a given time interval if for any $x(t)$ there exists a control $u(t)$.The system is also stabilizable if it is controllable, even if it is not completely stable. Similarly, the above system, with accompanying output equation as in Equation (2.1.2), is completely observable, for all t within the interval $[t_0, T_f]$, if for any output $y(t)$ there exists a state $x(t)$. The system is detectable if it is observable, or stable. Furthermore, there exit unique system solutions if the systems are both stabilizable and detectable.

3 Model Development

3.1 Population (Labour) growth dynamics

Suppose that the population (labour), $L_1(t)$, grows at an exponential rate, *n*, where, $0 < n < 1$, that is, in accordance with the Malthusian population growth. On the other hand, assume that the population, $L_2(t)$, is governed by a simple logistic growth model with a carrying capacity $\frac{1}{\sigma} > 0$, then

$$\frac{d}{dt} L_1(t) = nL_1(t) = f_1(L_1(t); n) \tag{3.1.1}$$

and

$$\frac{d}{dt} L_2(t) = n\big(1 - \sigma L_2(t)\big)L_2(t) = f_2(L_2(t); n, \sigma). \tag{3.1.2}$$

Per Equation (3.1.1), the equilibrium or critical value, L_e, of L_1 is $L_e = 0$. But $\frac{d}{dt} L_1(t) > 0$ for any initial value $L_0 > L_e = 0$, implies that $L_1(t) \to \infty$ as $t \to \infty$, for $0 < n < 1$. That is, since $f_1(L_e) = 0$ and $f_1^{'}(L_e) = n > 0$, for all $0 < n < 1$, $L_e = 0$ is an unstable equilibrium, a source.[1]

Similarly, from Equation (3.1.2), we obtain two equilibria, $L_{e_1} = 0$ and $L_{e_2} = \frac{1}{\sigma} > 0$. Moreover, for any initial population L_0 such that $L_{e_1} < L_0 < L_{e_2}$, $L_2(t) > 0$ and $L_2(t) \to L_{e_2}$ as $t \to \infty$, and thus we have $0 < L_2(t) < \frac{1}{\sigma}$, implying that $L_2(t)$ is bounded for all $t \geq 0$ and for $0 < n < 1$. Even for any $L_0 > L_{e_2}$, $L_2(t)$ decays gradually to L_{e_2} over time. Thus $L_{e_1} = 0$ is a source, whereas $L_{e_2} = \frac{1}{\sigma}$ is a stable equilibrium, a sink. From Equation (3.1.2), $L_2(t)$ and $L_1(t)$ are one and the same trajectory, when $\sigma = 0$. Thus $L_2(t)$ bifurcates when $\sigma = 0$. $L_2(t)$ bifurcates again when $\sigma = 1$, as the trajectory tends constant over time at $\sigma = 1$, but decays to zero over time when $\sigma > 1$, for $0 < n < 1$, and for all $t \geq 0$.

[1] *Additionally,* $f_1'(L_e) = n < 0$, *for all* $n < 0$. *Hence,* $n = 0$ *is a bifurcation value, and in contrast with the earlier discussion,* $L_1(t) \to 0$ *as* $t \to \infty$ *when* $n < 0$, *for* $L_0 > 0$.

Now, solving the two equations above, assuming our starting time $t_0 = 0$, we respectively obtain

$$L_1(t) = L_0 e^{nt} = e^{nt} \tag{3.1.3}$$

$$L_2(t) = \frac{L_0 e^{nt}}{1+\sigma L_0(e^{nt}-1)} = \frac{e^{nt}}{1+\sigma(e^{nt}-1)} \tag{3.1.4}$$

where for simplicity, and without any loss of generality, L_0 has been standardized to unity. However, from Equations (3.1.3) and (3.1.4), granted that $\sigma \geq 0$ and $0 < n < 1$, then for all $t \geq 0$, we obtain

$$\frac{L_1'(t)}{L_1(t)} = n \text{ and } \quad \frac{L_2'(t)}{L_2(t)} = \frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)} \tag{3.1.5}$$

$$\frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)} = \frac{n}{1+\frac{\sigma}{1-\sigma}e^{nt}} \leq n. \tag{3.1.6}$$

But $1 - \sigma + \sigma e^{nt} = 1 + \sigma\left(nt + \frac{n^2t^2}{2} + \cdots\right) \geq 1$, with equality only when $\sigma = 0$. Thus

$$\frac{e^{nt}}{1+\sigma(e^{nt}-1)} = \frac{e^{nt}}{(1-\sigma)+\sigma e^{nt}} \leq e^{nt}. \tag{3.1.7}$$

3.1.1 Sensitivity analysis on population growth dynamics

Suppose $L_1(t)$ and $L_2(t)$ are put together as $L(t)$ and defined as given in $L_2(t)$, that is

$$L(t) = \frac{e^{nt}}{1+\sigma(e^{nt}-1)}.$$

Then for all $t > 0$, we obtain the following partial derivatives

$$\frac{\partial L}{\partial \sigma} = -\frac{(e^{nt}-1)e^{nt}}{[1+\sigma(e^{nt}-1)]^2} < 0 \tag{3.1.8}$$

$$\frac{\partial L}{\partial n} = \frac{(1-\sigma)te^{nt}}{[1+\sigma(e^{nt}-1)]^2}\begin{cases} > 0 & \text{for } 0 \leq \sigma < 1 \\ = 0 & \text{for } \sigma = 1 \\ < 0 & \text{for } \sigma > 1 \end{cases} \tag{3.1.9}$$

and

$$\frac{\partial L}{\partial t} = \frac{(1-\sigma)ne^{nt}}{[1+\sigma(e^{nt}-1)]^2}\begin{cases} > 0 & \text{for } 0 \leq \sigma < 1 \\ = 0 & \text{for } \sigma = 1. \\ < 0 & \text{for } \sigma > 1 \end{cases} \tag{3.1.10}$$

Equation (3.1.8) shows that the population, $L(t)$, is a decreasing function of the parameter σ. Thus as the population growth becomes logistic, the lower the growth rate becomes, and by extension, the lower the population in relation to the one which grows exponentially, all things being equal. Equations (3.1.9) and (3.1.10) also suggest that the population, $L(t)$, is a increasing function of the parameter n, and of time, t, whenever $0 \leq \sigma < 1$. At the same time, $L(t)$ is a decreasing function of n and t when $\sigma > 1$, and it decays down to zero in the limit, and the higher the value σ the faster this is. Thus $\sigma > 1$ is a recipe for population extinction, and hence, undesirable. However, $L(t)$ becomes a constant function of n and t when $\sigma = 1$. From these three equations, it is inferably clear that, for all $t > 0$, $L(t)$ grows the fastest whenever $\sigma = 0$, that is when the growth dynamics of population is exponential. These results confirm an earlier observations made.

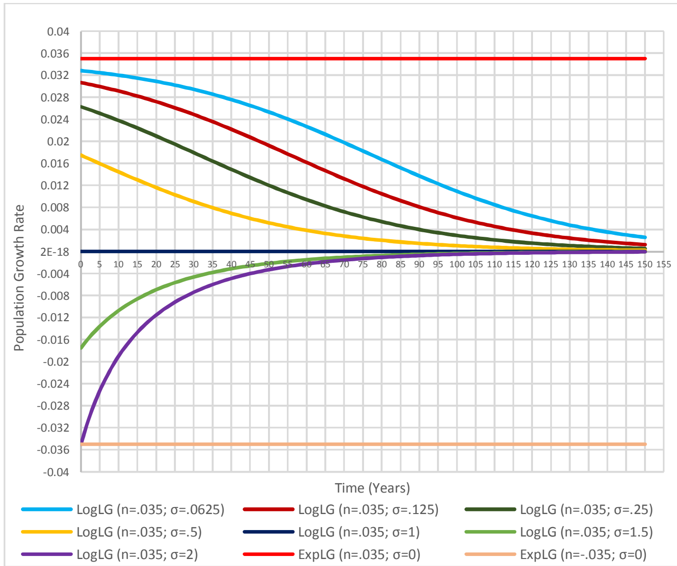

Fig. 3.1.1. Population growth dynamics over time for varying σ values

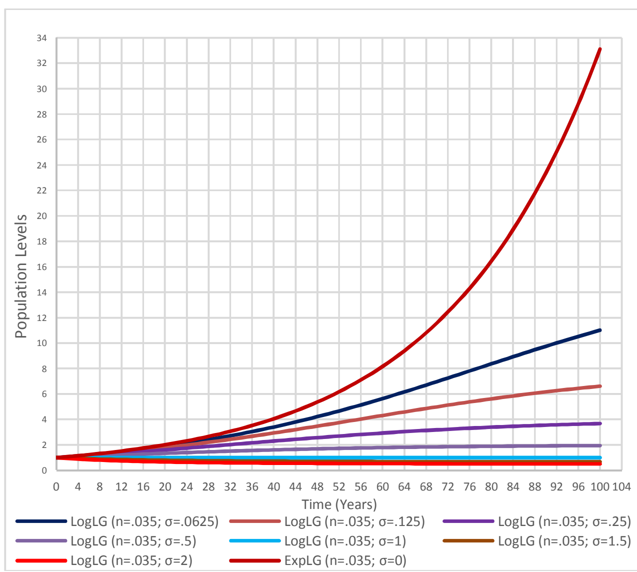

Fig. 3.1.2. Population dynamics over time for varying σ values

From the above, a population that exhibits logistic growth is consistently dominated by or equal to that which exhibits exponential growth. Fig. 3.1.1 above demonstrates this. Moreover, the higher the value of σ, the lower the population, and vice versa. Invariably, in computing real per capita income over time, the economy driven by logistic population growth would be sharing gross domestic product amongst much fewer people, in sharp contrast to the economy driven by exponential population growth. Hence, all things being equal, real per capita income in the economy driven by logistic population growth is most likely to be higher than in the economy driven by exponential population growth. The expositions in (3.1.6) to (3.1.10), together with its simulation plot in Fig. 3.1.1, and Fig. 3.1.2, collectively and individually, further confirm(s) this.

Table 3.1.1 shows the average population growth of some selected economies over a period, which tend to give credence to the population growth dynamics espoused above (data source: [23]). It can be inferred that all the lower income economies covered here experience exponential population growth. The lower middle income economies shown in the table also tend to indicate exponential population growth, except probably Indonesia whose data does not clearly show this. Gabon which is amongst the league of upper middle economies seems to experience exponential population growth. However, the population growth dynamics of higher income and higher middle income economies mostly tend to be logistic, as indicated by the data in the table. The plots of the actual population figures lend a lot more credence and clarity to the above.

Table 3.1.1: Population Growth Dynamics for Selected Countries Across Economics

Economy	Country	Average Population Growth Rate for Some Selected Economies (Percentage)						
		1961-1970	1971-1980	1981-1990	1991-2000	2001-2010	2011-2015	1961-2015
Higher Income	USA	1.106	0.904	0.846	1.098	0.824	0.586	1.019
	Belgium	0.502	0.191	0.109	0.244	0.577	0.362	0.370
	S. Korea	2.267	1.490	1.022	0.826	0.425	0.341	1.236
	Australia	1.781	1.280	1.348	1.032	1.273	1.265	1.501
Upper Middle Income	Algeria	2.465	2.560	2.653	1.620	1.325	1.557	2.292
	Gabon	1.587	1.939	2.449	2.326	2.654	2.607	2.471
	Brazil	2.538	2.190	1.858	1.404	1.078	0.721	1.878
	China	2.168	1.553	1.338	0.932	0.506	0.400	1.337
Lower Middle Income	India	1.905	2.096	2.009	1.718	1.394	1.000	1.928
	Indonesia	2.451	2.263	1.855	1.371	1.235	0.994	1.932
	Nigeria	1.979	2.521	2.359	2.279	2.380	2.156	2.528
	Ghana	2.273	2.039	2.782	2.332	2.355	1.887	2.560
Lower Income	S. Leone	1.459	2.056	2.270	0.512	3.144	1.824	2.083
	Gambia	1.731	2.763	3.899	2.613	2.940	2.624	3.073
	Ethiopia	2.284	1.890	2.903	2.936	2.503	2.093	2.733
	Benin	1.662	2.265	2.725	2.977	2.857	2.156	2.735

Source: World Bank Economic Database

3.2 Optimal control of closed economic growth model

Consider an economy in which income, $Y(t)$, is either expended on consumption, $C(t)$, or put into an investment, $I(t)$ [7, 8, 9, 10, 17]. That is,

$$Y(t) = C(t) + I(t). \tag{3.2.1}$$

Assume the only factors of production are labour, L, and capital, K. Hence, $Y(t) = Y\big(K(t), L(t)\big)$, and the map $Y: R_+^2 \longmapsto R_+$ displays constant returns to scale, such that

$$Y_K > 0,\ Y_L > 0 \quad \text{and} \qquad Y_{KK} < 0,\ Y_{LL} < 0.$$

Moreover, Y satisfies the Inada conditions as in [10] thus
$\lim_{K\to 0} Y_K = \infty$, $\lim_{K\to\infty} Y_K = 0, \forall L > 0$; and $\lim_{L\to 0} Y_L = \infty$, $\lim_{L\to\infty} Y_L = 0, \forall K > 0$.

If μ is the rate of depreciation in $K(t)$, then in its simplest form, economic theory [10, 17, 24, 25, 26, 27, 28, 29] states that

$$I(t) = \dot{K}(t) + \mu K(t). \tag{3.2.2}$$

Let $y = Y/L$, $c = C/L$, and $k = K/L$, and assume a production function of unit homogeneity. Then

$$y(t) = f(k(t)) = c(t) + \frac{1}{L(t)}\left(\frac{dK(t)}{dt} + \mu K(t)\right) = c(t) + \mu k(t) + \frac{1}{L(t)}\frac{dK(t)}{dt}. \quad (3.2.3)$$

Differentiating $k = K/L$ with respect to t, and using results in Section 3.1, then our state equation is

$$\dot{k}(t) = f(k(t)) - \left(\mu + \frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)}\right)k(t) - c(t) = g(k(t), c(t), t) \quad (3.2.4)$$

where $k(t)$ is the state variable, $c(t)$ is the control variable, and n, σ and μ, are system parameters.

3.2.1 The utility and welfare functional

The utility, $U(t)$, derived from the consumption of goods and services, $C(t)$, is given by

$$U(t) = U(C(t)), \quad (3.2.5)$$

with the property $U'(C(t)) > 0$ and $U''(C(t)) < 0$ [10, 27, 28]. The welfare functional, W, above becomes

$$W(C) = \int_{t_0}^{T_f} U(C(\tau))d\tau. \quad (3.2.6)$$

The utility function per capita is $u(c(t)$. Let γ denote the discount rate. Then for $\gamma - n > 0$,

$$W(c) = \int_{t_0}^{T_f} \frac{e^{-(\gamma-n)\tau}u(c(\tau))}{1+\sigma(e^{n\tau}-1)}d\tau, \qquad T_f \geq t_0. \quad (3.2.7)$$

3.2.2 The optimal control problem

Our problem simplifies to finding the control [17, 25, 27, 28], c, which maximizes the objective functional

$$W(c) = \int_{t_0}^{T_f} \frac{e^{-(\gamma-n)\tau}u(c(\tau))}{1+\sigma(e^{n\tau}-)}d\tau, \qquad T_f \geq t_0 \geq 0$$

subject to $\dot{k}(t) = f(k(t)) - \left(\mu + \frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)}\right)k(t) - c(t)$

for $k(t_0) = k_0 \geq 0$, $k(T_f) = k_{T_f} \geq 0$

with output $y(t) = f(k(t))$.

3.3 System Equations

Assume Y is linear in K and L, that is, Y is defined by $Y(K(t), L(t)) = \alpha K(t) + \beta L(t)$. Then from the above, $f(k(t)) = \beta + \alpha k(t)$, where, $0 < \alpha < 1$, and $0 < \beta < 1$.

$$\dot{k}(t) = \left(\alpha - \mu - \frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)}\right)k(t) - c_1(t) \quad (3.3.1)$$

and $$z(t) = \alpha k(t) \quad (3.3.2)$$

where $z(t) = y(t) - \beta$ and $c_1(t) = c(t) - \beta$.

3.4 Local stability, controllability and observability of the systems

Per Equation (3.3.1) the system is locally controllable, since, clearly, given any state $k(t)$, there exists a control $c(t)$ which drives the system from one state to the other. It is asymptotically stable if and only if $\alpha < \mu + \frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)}$, for all $t \geq 0$. However, since this is not guaranteed, the system may not generally be stable. However, given that it is completely controllable suggests it is stabilizable. From Equation (3.3.2), given that for any output $y(t)$ there exists a state $k(t)$, the system is observable, and thus, reconstructible, and subsequently, detectable [18, 21]. The systems being both stabilizable and detectable suggest that there exists unique control, $c(t)$, within the admissible control set, with associated state trajectory $k(t)$, which solve the problem.

3.5 Sensitivity and bifurcation analyses of the systems

From Equation (3.2.3), using the fact that at the equilibrium $I(t) = S(t) = sY(t)$, where $S(t)$ is the aggregate savings in the economy at any time t, and s is the propensity to save or the savings rate, we have $c(t) = (1-s)y(t) \approx (1-s)\alpha k(t)$. Thus Equation (3.2.4) becomes

$$\dot{k}(t) = \left(s\alpha - \mu - \frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)}\right)k(t) = g_{\text{para}}(k(t), t). \tag{3.5.1}$$

Thus the equilibrium value, $k^e = 0$, is given by $\dot{k} = 0$. As per the argument in Section 3.1, $k(t) = k^e$ is a source, and for any initial value $k(t_0) > k^e$, which is in the domain of interest. Thus $k(t) \to \infty$ as $t \to \infty$. But at the equilibrium, $s = \alpha$, $0 < \alpha < 1$, and thus given $g_{\text{para}}(k^e) = 0$ and

$$g'_{\text{para}}(k^e) = \alpha^2 - \mu - \frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)} \begin{cases} > 0, & \text{for } \alpha^2 > \mu + \frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)} \\ = 0, & \text{for } \alpha^2 = \mu + \frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)} \\ < 0, & \text{for } \alpha^2 < \mu + \frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)} \end{cases} \tag{3.5.2}$$

and thus $\alpha = \left(\mu + \frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)}\right)^{\frac{1}{2}}$ is a bifurcation value. Similarly, $\dot{k}(t) < 0$, if $\alpha \leq \mu + \frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)}$. But $\dot{k}(t) \gtrless 0$ whenever $\alpha > \mu + \frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)}$, with a driving force $c(t) = (1-s)\alpha k(t) < k(t)$. Invariably, $k(t)$ bifurcates at $\alpha = \mu + \frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)}$. As α varies through $\mu + \frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)}$, from below, growth in $k(t)$ changes from starting a bit faster and slowing down to starting slower and picking up over time. This is reversed if $\frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)} + \mu < \alpha < \left(\frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)} + \mu\right)^{\frac{1}{2}}$, and tends increasingly faster still whenever $\alpha > \left(\frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)} + \mu\right)^{\frac{1}{2}}$. These characteristic features translate into the corresponding trajectories of $y(t)$.

Subsequently, from $y(t) = f(k(t))$, we obtain

$$\frac{\dot{y}(t)}{y(t)} = \frac{f'(k(t))\cdot k(t)}{f(k(t))} \cdot \frac{\dot{k}(t)}{k(t)} = \varepsilon_k(k(t))\frac{\dot{k}(t)}{k(t)} = \alpha\frac{\dot{k}(t)}{k(t)} \tag{3.5.3}$$

$$\Rightarrow \qquad \frac{\dot{y}(t)}{y(t)} = \alpha\left(s\alpha - \mu - \frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)}\right) = m(t; \alpha, s, \mu, n, \sigma) \tag{3.5.4}$$

using Equation (3.5.1), where $\alpha = \varepsilon_k(k(t)) = \frac{f'(k(t))k(t)}{f(k(t))} \in (0,1)$ is the elasticity of the production function f, which also measures the share of capital in the production mix [10].

To ensure a sustained rise in real per capita income, and hence, economic development, or exit from the poverty trap, if it exists, then in a typical economy described by our system, the policy objective should be having the right hand side of Equation (3.5.4) to be positive. That is, we should have

$$s\alpha > \mu + \frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)}, \qquad \text{for all } t \geq 0 \tag{3.5.5}$$

$$\text{or } \alpha > \left[\mu + \frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)}\right]^{\frac{1}{2}}, \text{ for all } t \geq 0 \tag{3.5.6}$$

given that $\alpha \in (0, 1)$, and assuming $s = \alpha$ at the equilibrium, as used earlier. The more positively large it is, the faster the growth in real per capita income. Real per capita income stagnates if the said expression is zero, whereas there is retrogression where it is negative. Whenever $\sigma = 0$, the right hand side of (3.5.4) is more likely be marginally greater than or equal to or less than zero, and hence, the more likely the growth in $y(t)$ becomes slower, or stagnates, or slows down the more, accordingly. However, when $\sigma > 0$, the right hand side of (3.5.4) is more likely to be far greater than or equal to marginally less than zero. Thus growth in $y(t)$ is more likely to be correspondingly faster, or constant, or marginally slow down, all things being equal. The greater the value of σ, the better.

From Equation (3.5.4), and within the defined set of values for each parameter, we have

$$\frac{\partial m}{\partial \sigma} = \frac{n\alpha e^{nt}}{[1+\sigma(e^{nt}-1)]^2} > 0 \tag{3.5.7}$$

$$\frac{\partial m}{\partial t} = \frac{n^2(1-\sigma)\sigma\alpha e^{nt}}{[1+\sigma(e^{nt}-1)]^2} \begin{cases} > 0, & \text{for } 0 < \sigma < 1 \\ = 0, & \text{for } \sigma = 0\,, 1 \\ < 0, & \text{for } \sigma > 1 \end{cases} \tag{3.5.8}$$

for all $t \geq 0$. From (3.5.7), and hence, Equation (3.5.4), that, as ε increases so does the trajectory of $y(t)$ rise over time, and vice versa. However, using (3.5.8) and Equation (3.5.4) suggest that the growth in $y(t)$ rises when $0 < \sigma < 1$, stagnates whenever $\sigma = 0$, and slows down when $\sigma > 1$. Similarly,

$$\frac{\partial m}{\partial \alpha} = 2s\alpha - \mu - \frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)} \tag{3.5.9}$$

$$\frac{\partial m}{\partial \mu} = -\alpha < 0 \tag{3.5.10}$$

$$\frac{\partial m}{\partial n} = -\frac{[\sigma(1-nt)e^{nt}+1-\sigma](1-\sigma)\alpha}{[1+\sigma(e^{nt}-1)]^2} \begin{cases} < 0 & \text{for } 0 \leq \sigma < 1 \\ = 0 & \text{for } \sigma = 1. \\ > 0 & \text{for } \sigma > 1 \end{cases} \tag{3.5.11}$$

Equations (3.5.9) and (3.5.4) consequently suggest that growth in $y(t)$ shoots up as α is increased, and more pronounced when $\sigma > 0$. The converse is equally true. On the other hand, we can deduce from (3.5.10) and (3.5.4) that increases in μ negatively impact on the growth prospects in $y(t)$, and vice versa. The same analysis holds in respect of n whenever $0 \leq \sigma < 1$. However, for $\sigma > 1$, higher values of n generate higher time trajectories of $y(t)$, except that $\sigma > 1$ is inimical to the very sustenance of population. But n has no effect on the time performance of real per capita GDP whenever $\sigma = 1$.

3.6 The Maximum Sustainable Population Growth Analysis

Consider a simple Malthusian economy in which the only factors of production are natural resources, $R(t)$, and labour, $L(t)$. Assume that the aggregate production function is linear in $R(t)$ and $L(t)$, akin to what has been used above, that is, $Y(t) = \varepsilon R(t) + (1-\varepsilon)L(t)$, where $0 < \varepsilon < 1$ is the elasticity of $Y(t)$ in respect of

$R(t)$. Assume further that μ_r, s_r and MP_r are respectively the rates of depreciation in-, savings in-, and the marginal product $r(t)$, $r(t) = \frac{R(t)}{L(t)}$, then

$$\frac{\dot{y}(t)}{y(t)} = \frac{\varepsilon\dot{r}(t)}{\varepsilon r(t)+\beta} \approx \frac{\dot{r}(t)}{r(t)} = s_r MP_r - \mu_r - \frac{\dot{L}(t)}{L(t)}. \tag{3.6.1}$$

At the equilibrium income per head, the maximum sustainable population growth, MSPG, defining the fastest rate at which population can grow without impacting negatively on $y(t)$, is given by

$$MSPG = s_r MP_r - \mu_r. \tag{3.6.2}$$

Equation (3.6.1) suggests that when s_r and or MP_r increase(s) then $y(t)$ rises. The converse is also true. However, whenever μ_r and or population growth go(es) up, then $y(t)$ declines, and vice versa. If μ_r and population growth decline at the same time as s_r and MP_r also rise, growth in $y(t)$ becomes much faster. But s_r and MP_r enhance the value of MSPG whereas μ_r inversely impacts on MSPG.

With the presence of technology, labour augmenting technology, then Equation (3.6.1) becomes

$$\frac{\dot{y}(t)}{y(t)} \approx (1-\varepsilon)\frac{\dot{A}(t)}{A(t)} + s_{\hat{r}} MP_{\hat{r}} - \varepsilon\left[\mu_{\hat{r}} + \frac{\dot{L}(t)}{L(t)}\right] \tag{3.6.3}$$

$$\Rightarrow \qquad MSPG = \frac{1}{\varepsilon}\left[(1-\varepsilon)\frac{\dot{A}(t)}{A(t)} + s_{\hat{r}} MP_{\hat{r}} - \varepsilon\mu_{\hat{r}}\right] \tag{3.6.4}$$

where $A(t)$ is technology and $\hat{r}(t) = \frac{R(t)}{A(t)L(t)}$ and $s_{\hat{r}}$, $\mu_{\hat{r}}$ and $MP_{\hat{r}}$ are as similarly defined earlier. From Equations (3.6.3) and (3.6.4), the effects of population growth, $s_{\hat{r}}$, $\mu_{\hat{r}}$ and $MP_{\hat{r}}$ on the performance of $y(t)$, as well as on MSPG, are same as discussed earlier. In addition, whereas technological growth impacts positively on the growth of $y(t)$ and MSPG, ε inversely affects the MSPG, and hence, the performance of $y(t)$. Thus as economic activities become predominantly natural-resources-based, the economy's capacity shrinks or $y(t)$ retrogresses, as MSPG declines. The converse is similarly true.

Fig. 3.6.1 below depicts a basic interface between income and MSPG, showing two equilibria points M and G, Malthusian and growth respectively. Whereas M is a sink, a stable equilibrium point, G is a source, an unstable equilibrium point. Consequently, whenever $y(t) < y_M$, $y(t)$ grows till it reaches y_M , the Malthusian equilibrium real income per capita. If $y_M < y(t) < y_G$, then $y(t)$ declines till it gets back to y_M. But when $y(t) > y_G$, growth in $y(t)$ persists. As the economy experiences increments in $A(t)$ or $s_{\hat{r}}$ or $MP_{\hat{r}}$ or a combination of these over time, or a decrements in ε or $\mu_{\hat{r}}$ or population growth or a combination of these, or both conditions, the MSPG shoots up, else it may fall.

If the upsurge in MSPG is ample enough to cause it to move above the cap of the MSPG-income curve, that is, above the peak of any actual observable population growth, then capacity is created within the economy for economic take-off, else it ends up increasing the population growth rate concurrently. See Fig. 3.6.1 adjacent. The coming in of additional factors of aggregate production may help in this direction, as exemplified in Equation (3.6.5) below.

Thus if new natural resources, $R_i(t)$, capitals, $K_j(t)$, and some other variable factors (like trade and policy options), $X_l(t)$, are discovered and utilized, with corresponding savings rates, marginal products, such as $s_{\hat{r}_i}$, and $MP_{\hat{r}_i}$, and so on, then a generalised form of Equation (3.6.4) becomes

$$MSPG = \frac{1}{\eta}\left[(1-\eta)\frac{\dot{A}(t)}{A(t)} + \sum_i s_{\hat{r}_i} MP_{\hat{r}_i} + \sum_j s_{\hat{k}_j} MP_{\hat{k}_j} + \sum_l s_{\hat{x}_l} MP_{\hat{x}_l} - \sum_\tau \epsilon_\tau \mu_\tau\right]. \tag{3.6.5}$$

$\sum_\tau \epsilon_\tau \mu_\tau$ takes care of all the depreciation rates multiplied by relevant factor shares, and η is the sum of all the factor shares. The factor shares of $R_i(t)$, $K_j(t)$ and $X_l(t)$ are respectively ε_i, α_j and β_l. The terms $\hat{r}_i$, $\hat{k}_j$ and $\hat{x}_l$ are the analogies of $\hat{r}$ respectively in terms of $R_i(t)$, $K_j(t)$ and $X_l(t)$. The sums of α_j and β_l form part of η, and inversely impact on MSPG, as per Equation (3.6.5). But Equations (3.3.2) and (3.5.10) suggest that α, of which kind are the α_j, and hence, β_l, positively impacts on the performance of $y(t)$. Consequently, to exit the Malthusian trap is to pursue measures that enhance technological growth, savings rates and marginal products per (effective) labour of the various factors of production, whilst maintaining lower depreciation rates as well as minimal shares of natural resources in the aggregate production function.

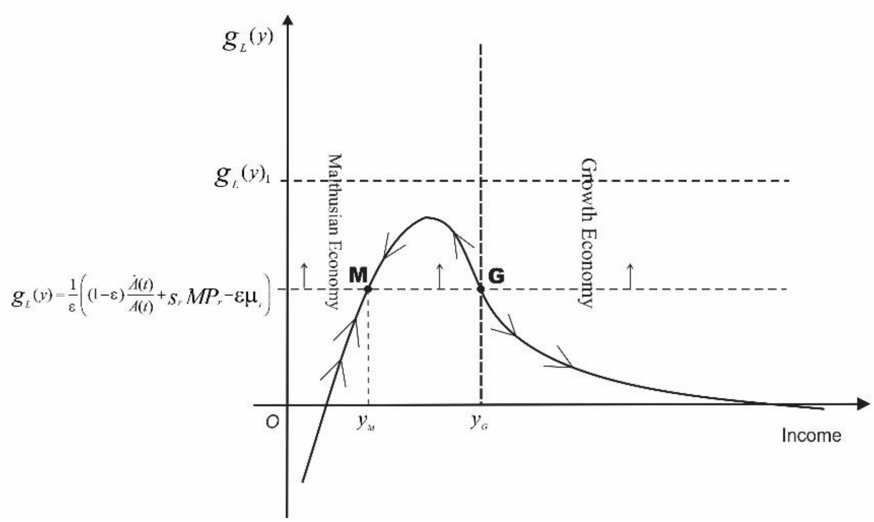

Fig. 3.6.1. MSPG-real per capita income dynamics

Unlike the results here, in [30], besides technological growth, the other drivers for real per capita GPD growth is the sum of the products of the growths with respect to the various factors of production and related factor shares (besides labour), with labour (population) growth and its associated factor share being the drawback elements.

3.7 Hamilton-Pontryagin equations for the systems

The associated Hamiltonian function, H, for systems is given by

$$H(k(t), c(t), \lambda(t), t) = \frac{e^{-(\gamma-n)t}u(c(t))}{1+\sigma(e^{nt}-1)} + \lambda(t)\left[\left(\alpha - \mu - \frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)}\right)k(t) - c_1(t)\right]. \tag{3.7.1}$$

Using a logarithmic utility functional, $u(c(t)) = \ln c(t)$, then Equations (2.2.3) to (2.2.5), give

$$c(t) = \frac{e^{-(\gamma-n)t}}{[1+\sigma(e^{nt}-1)]\lambda(t)} \tag{3.7.2}$$

and $$\dot{\lambda}(t) = \left(\mu + \frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)} - \alpha\right)\lambda(t) \tag{3.7.3}$$

$$\dot{k}(t) = \left(\alpha - \mu - \frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)}\right)k(t) - c_1(t) \tag{3.7.4}$$

for $k(0) = k_0 > 0, \;\; k(T_f) = k_{T_f} \geq 0.$ (3.7.5)

3.8 Analytical solutions to the Hamilton-Pontryagin equations

From Equation (3.7.3), we obtain

$$\lambda(t) = \frac{\lambda_0}{1+\sigma(e^{nt}-1)} e^{(n+\mu-\alpha)t}, \tag{3.8.1}$$

where λ_0 is the required initial value of λ. Hence, Equation (3.7.2) becomes

$$c(t) = \frac{e^{-(\gamma-n)t}}{\lambda_0} e^{(\alpha-n-\mu)t} = \frac{1}{\lambda_0} e^{(\alpha-\mu-\gamma)t}. \tag{3.8.2}$$

From Equations (3.7.2) and (3.7.3), we may also obtain the equation of motion of $c(t)$ as

$$\dot{c}(t) = (\alpha - \mu - \gamma)c(t). \tag{3.8.3}$$

Now, putting Equation (3.8.2) into the state equation, that is, Equation (3.7.4), and solving, we obtain

$$k(t) = [1 + \sigma(e^{nt} - 1)] e^{(\alpha-\mu-n)t} \left[k_0 + \zeta(t) - \frac{1}{\lambda_0} h(t)\right] \tag{3.8.4}$$

where $h(t) = \int_0^t \frac{e^{-(\gamma-n)\tau}}{1+\sigma(e^{n\tau}-1)} d\tau$ and $\zeta(t) = \beta \int_0^t \frac{e^{(\mu+n-\alpha)\tau}}{1+\sigma(e^{n\tau}-1)} d\tau$. Using the terminal condition gives,

$$\lambda(t) = \frac{f_3(T_f)}{f_4(T_f)} \cdot \frac{e^{-\omega t}}{f_1(t)} \tag{3.8.5}$$

$$c(t) = \frac{f_4(T_f)}{f_3(T_f)} e^{(\alpha-\mu-\gamma)t} \tag{3.8.6}$$

$$k(t) = q(t)e^{\omega t} \tag{3.8.7}$$

$$\text{and} \quad y(t) = \beta + \alpha q(t) e^{\omega t} \tag{3.8.8}$$

for $\omega = \alpha - \mu - n$, $f_1(t) = 1 + \sigma(e^{nt} - 1)$, $f_2(t) = k_{T_f} e^{-\omega t}$, $f_3(t) = f_1(t)h(t)$, $\gamma - n > 0$, $\gamma > 0$, $q(t) = \left(\frac{\left[[h(T_f)-h(t)]k_0 + h(T_f)\zeta(t) - \zeta(T_f)h(t)\right] f_1(T_f) + f_2(T_f)h(t)}{f_3(T_f)}\right) f_1(t)$ and $f_4(t) = (k_0 + \zeta(t)) f_1(t) - f_2(t)$.

Assuming we introduce technology into our system, such that technology grows exponentially at a, $0 < a < 1$, then Equations (3.3.1), (3.5.1) and (3.5.4) respectively becomes

$$\dot{\hat{k}}(t) = \left(\alpha - a - \mu - \frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)}\right) \hat{k}(t) - \hat{c}(t) \tag{3.8.9}$$

$$\dot{\hat{k}}(t) = \left(s\alpha - a - \mu - \frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)}\right) \hat{k}(t) \tag{3.8.10}$$

$$\frac{\dot{y}(t)}{y(t)} = (1-\alpha)a + \alpha\left(s\alpha - \mu - \frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)}\right) = m(t; a, \alpha, s, \mu, n, \sigma). \tag{3.8.11}$$

$$\Rightarrow \quad \frac{\partial m}{\partial a} = 1 - \alpha > 0 \tag{3.8.12}$$

and hence, growth in $y(t)$ steps up whenever a increases, all things being equal. The reverse is also true. The other results obtained in Section 3.5 remain similarly the same here. Solving gives

$$\lambda(t) = \frac{f_3(T_f)}{f_7(T_f)} \cdot \frac{e^{-\omega_1 t}}{f_1(t)} \tag{3.8.13}$$

$$c(t) = \frac{A_0 f_7(T_f)}{f_3(T_f)} e^{(\alpha-\mu-\gamma)t} \tag{3.8.14}$$

$$k(t) = A_0 q_1(t) e^{\omega t} \tag{3.8.15}$$

$$y(t) = A_0\left[\beta f_3(T_f) + \alpha q_1(t) e^{\omega_1 t}\right] e^{at} \tag{3.8.16}$$

where $\omega_1 = \omega - a$, $f_6(t) = \hat{k}_{T_f} e^{-\omega_1 t}$, $f_7(t) = \left(\hat{k}_0 + \zeta_1(t)\right) f_1(t) - f_6(t)$, $\zeta_1(t) = \beta \int_0^t \frac{e^{-\omega_1 \tau}}{1+\sigma(e^{n\tau}-1)} d\tau$, $q_1(t) = \left(\frac{\left[[h(T_f)-h(t)]k_0 + h(T_f)\zeta_1(t) - \zeta_1(T_f)h(t)\right] f_1(T_f) + f_6(T_f)h(t)}{f_3(T_f)}\right) f_1(t)$.

3.9 Discussion

From Section 3.1, when a population exhibits logistic growth, its growth is moderated, and hence, generates much less numbers, in comparison with the one that exhibits exponential growth. Figs. 3.1.1 and 3.1.2 also attest to this. Thus, all things being equal, the time-performance of real per capita income is much better in the economy in which population growth is logistic. The greater the value of σ, the better, all things being equal. All graphs, especially Fig. 3.9.1, illustrate this.

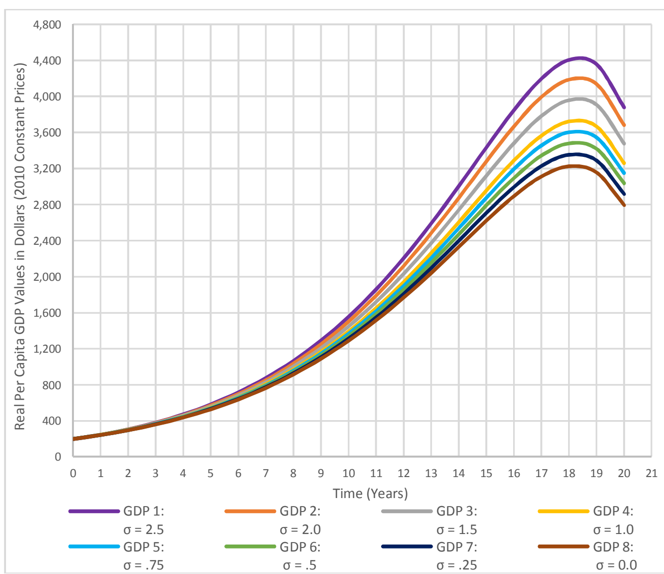

Fig. 3.9.1. Real per capita GDP trajectories for varying values of σ

3.9.1 The systems without technology

Equations (3.5.4) and (3.5.10), suggest that $\frac{\partial y}{\partial \alpha} > 0$, and together with (3.5.6) and (3.5.7), it follows that higher values of α ignites higher performance in $y(t)$ over time, all things being equal. The converse is also true. Fig. 3.9.2 gives credence to this. When the condition in (3.5.6) or (3.5.7) is accomplished, which is stepped up when σ is greater, then the faster $y(t)$ grows. The reverse is similarly true. The above is also true in respect of k_0. This is corroborated by Fig. 3.9.3 beneath.

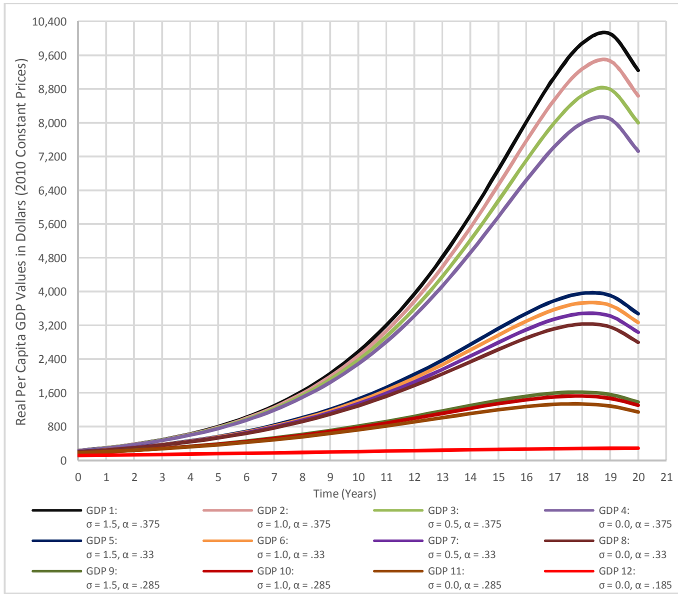

Fig. 3.9.2. Real per capita GDP trajectories for varying values of σ & α

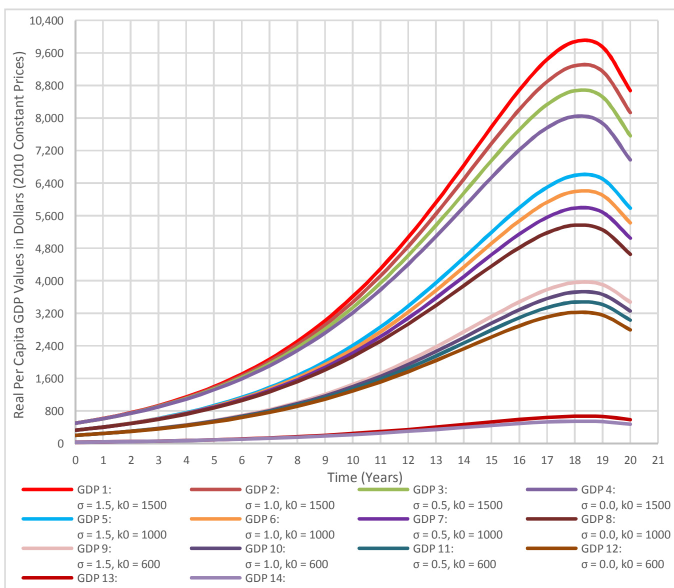

Fig. 3.9.3. Real per capita GDP trajectories for varying values of σ &

From (3.5.12) and (3.5.4), it is evident that $\frac{\partial y}{\partial n} < 0$, for all $t \geq 0$, irrespective of the system under consideration. Thus, all things being equal, decreasing the value of n, ignites higher time-values of real per capita GDP, and vice versa. This negative impact of population growth rate on real per capita GDP is minimized when $0 < \sigma < 1$, especially when σ is closer to 1, and eliminated completely when $\sigma = 1$. When $\sigma > 1$, higher values of n tend to generate higher time-trajectories of real per capita income, except that this is undesirable since $\sigma > 1$ is a recipe for a population extinction. Fig. 3.9.4 goes a long way to authenticate these. (Fig. 3.9.5 and Fig. 3.9.6 correspondingly seeks to illustrate the negative effect of μ, and γ, on the time-dynamics of real per capita GDP.)

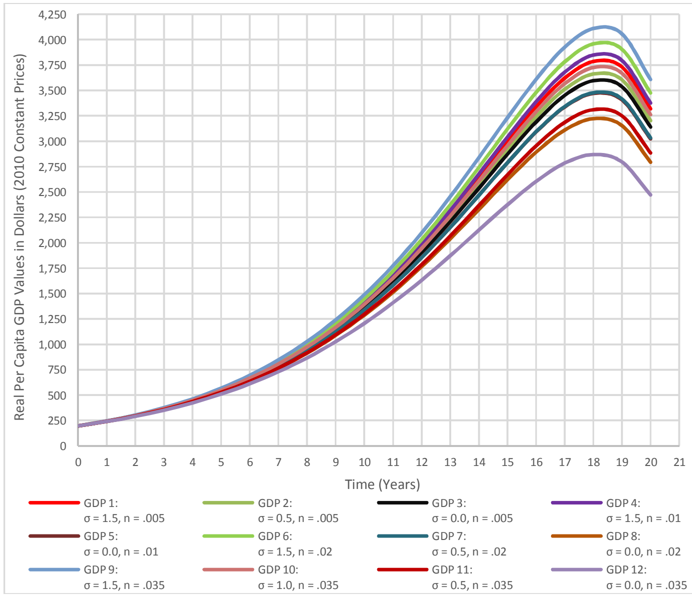

Fig. 3.9.4. Real per capita GDP trajectories for varying values of σ & n

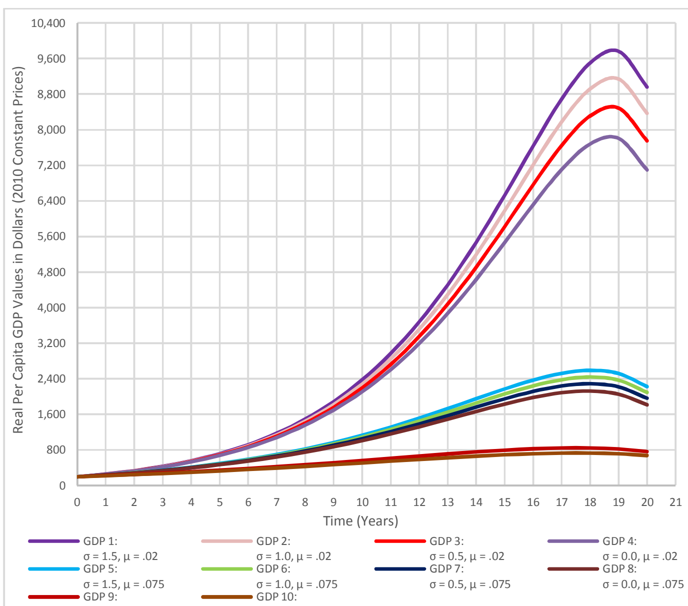

Fig. 3.9.5. Real per capita GDP trajectories for varying values of σ & μ

Subsequently, it is instructive to note that higher values of α and σ in combination with smaller values of n (as well as those of μ and γ), except when $0 \leq \sigma < 1$, is the right stimulus for a higher sustained growth in real per capita income dynamics over time, when the production function is linear.

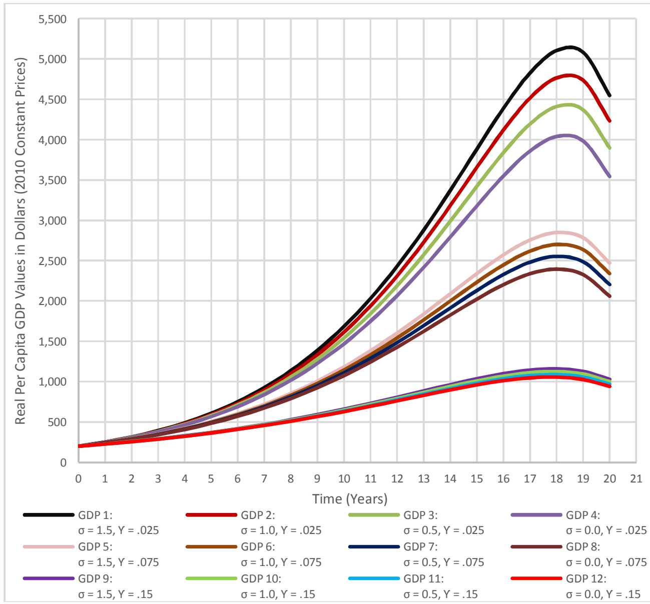

Fig. 3.9.6. Real per capita GDP trajectories for varying values of σ & γ

3.9.2 The system with technology

Equation (3.8.11), and (3.8.12) above suggest that $\frac{\partial y}{\partial a} > 0$, and hence, higher values of a generates higher performance in real per capita GDP over time, all things being equal. Fig. 3.9.7 amply depicts this phenomenon. Furthermore, comparing Equation (3.8.11) with Equation (3.5.4), it quite evident that a props up the growth prospects in $y(t)$, and thus a key proponent in helping escape the poverty or Malthusian trap, if it exists. The higher the values of σ, a α, the greater the growth rate is and the faster this is achieved. Fig. 3.9.7 re-affirms this. The impact of the other parameters on the time evolution of real per capita GDP is similar to the discussions in Section 3.9.1 above.

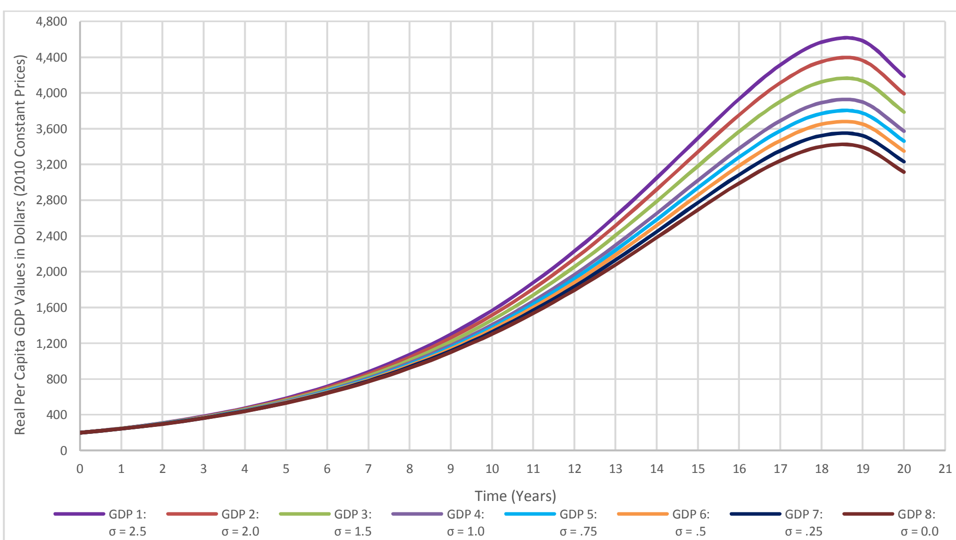

Fig. 3.9.7. Real per capita GDP trajectories for varying values of σ (Tech)

4 Conclusion

Generally, both models created above are not stable, not even at the equilibrium point. But they are each controllable, and thus, stabilizable, implying solutions are reachable. The systems are also observable, and thus detectable. Hence, there exist unique solutions to the economic growth problem in general. In spite of the fact that the models are overly simplified, their forecasts, in control, state and outputs, are reasonably good. The analyses suggest that the systems are sensitive to changes in the system parameters, of which includes the population dynamics parameter, σ,[2] plays a fundamental role.

The parameter σ largely determines the extent to which some system parameters impact on the time performance of real per capita income, $y(t)$. The models indicate that, to ensure faster growth in real per capita income, the population growth dynamics should be increasingly logistic, the share of capital in the aggregate production mix should be gradually higher, as well as the technological growth rate. Concurrently, the natural growth rate of population (discount and capital depreciation rates, as well) should be progressively lower, except when $\sigma > 1$, safe the stated caveat on it.

Additionally, higher technological growth, savings and marginal products per (effective) labour of the various factors of production, with lower depreciation rates and minimal shares of natural resource-based factors in the aggregate production mix help exit the Malthusian trap, if it exists.

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Competing Interests

Authors have declared that no competing interests exist.

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[2] Solow R. A Contribution to the theory of economic growth. Review of Economics and Statistics. 1956

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[2] *Note that in the referenced plots, as shown above, unless otherwise stated or a simulation is examining the effects of the perturbation in a particular parameter whereby the plots indicate the parameter values used, we have mostly used* $k_0 = \hat{k}_0 = 600$, $\alpha = 0.33$, $n = 0.02$, $\gamma = 0.045$, *as well as* $\mu = 0.05$ *and* $A_0 = 1$.

$\sigma = 0$	*the population (labour) grows exponentially at its natural growth* n *over time.*
$\sigma = 1$	*the population (labour) is static and do not grow at all or grows at a 0.00% over time.*
$\sigma > 1$	*the population (labour) declines or grows at a negative rate over time.*
$0 < \sigma < 1$	*the population (labour) grows but at a reducing rate over time.*

For $\sigma > 0$ *the population growth dynamics is said to be logistic. We assume that the natural growth rate of population,* n*, itself is non-negative.*

Though the model graphs created use hypothetical estimates of the model parameters to generate numerical solutions of the model and plot, these parameter estimates are actually calculated, or guarded by calculations from economic databases on different economies across different spectrum of economic standing. However, the estimates of μ *and* γ *are generally assumed. This notwithstanding, the import of the discourse is not affected. Thus the analysis only stops short of mapping economies to specific solution sets and graphs generated.*

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