Asian Research Journal of Mathematics
2(2): 1-11, 2017; Article no.ARJOM.30865

SCIENCEDOMAIN *international*
www.sciencedomain.org

SCIENCEDOMAIN

The Method of Multiple Time Scales and Finite Differences Method for the van der Pol Oscillator with Small Fractional Damping

Fadime Dal[1*]

[1]*Department of Mathematics, Ege University, Izmir, Turkey.*

Author's contribution

The sole author designed, analyzed and interpreted and prepared the manuscript.

Article Information

DOI: 10.9734/ARJOM/2017/30865
Editor(s):
(1) Igor Ya. Subbotin, Department of Mathematics, National University, LA, CA, USA.
Reviewers:
(1) Francisco Gomez, CENIDET, México.
(2) Youssri H. Youssri, Cairo University, Egypt.
Complete Peer review History: http://www.sciencedomain.org/review-history/17597

Review Article

Received: 5th December 2016
Accepted: 16th January 2017
Published: 24th January 2017

Abstract

In this paper, we consider the van der Pol oscillator with small fractional damping. To construct the approximate and numerical solutions of the equation, the method of multiple time scales and finite differences method are used, respectively. It is shown that the approximate solution is in good agreement with the numerical solution for weakly nonlinear case and the fractional derivative order has meaningful effects on natural frequencies of the oscillator.

Keywords: Fractional differential equation; Caputo fractional derivative; vibration; van der Pol equation; multiple time scale; finite difference.

NOMENCLATURES

x : *The dynamical variable*
t : *Independent variable denotes time*
ε : *Positive small parameter*
D^{α} : *Fractional-order differential operator in Caputo sense*
$\Gamma(\cdot)$: *The gamma function*
α : *Order of fractional derivative*

**Corresponding author: E-mail: fadimedal@hotmail.com;*

1 Introduction

Fractional calculus has been an important mathematical topic since 17th century. Although it has a long history, applications of the study have started in recent years. The researchers have been focused on applications of the topic only the area of physics and engineering. More recently, a new trend has been to investigate the control and dynamics of the fractional order dynamical systems [1-4]. In spite of the fact that traditional dynamic models are based on integer order differentiation and integration. Many real dynamic systems such as viscoelastic systems, dielectric polarization, electrode–electrolyte polarization, and electromagnetic waves [1-3] are better characterized using a non-integer order dynamic model based on fractional calculus or, differentiation or integration of non-integer order.

In this respect, the study of nonlinear oscillators is important in the development of the theory of dynamical systems. For example, a van der Pol system, which is a typical nonlinear chaotic system, has many interesting features and numerous applications. The Van der Pol oscillator (VPO) can be regarded as describing a mass–spring–damper system with a nonlinear position. This system depends on damping coefficient or, equivalently, an RLC electrical circuit with a negative-nonlinear resistor, and it is used for the design of various systems including biological ones, such as the heartbeats or the generation of action potentials by neurons, acoustic models, the radiation of mobile phones, and as a model of electrical systems [2-4].

Fractional calculus is a very important branch of mathematical analysis. There are extensive studies concerning fractional calculus from both theoretical and practical points of view.

It is well-known that the fractional calculus deals with derivatives and integrals to an arbitrary order (real or complex). The applications of fractional calculus are numerous in many fields. For example, several problems in mechanics (theory of viscoelasticity and viscoplasticity), (bio-)chemistry (modeling of polymers and proteins), medicine (modeling of human tissue under mechanical loads) electrical engineering (transmission of ultrasound waves), and other problems can be modeled by fractional differential equations. Analytical solutions of fractional-differential equations are not always available, and therefore, it is an important matter to obtain numerical solutions for such equations via several techniques. In this respect, a great number of researchers have considerable interests in investigating numerically various types of fractional differential equations (FDEs). For example [A,B,C,D,E]

[A] A Novel Operational Matrix of Caputo Fractional Derivatives of Fibonacci Polynomials: Spectral Solutions of Fractional Differential Equations. Entropy 18 (10), 345.

[B] Spectral Solutions for Fractional Differential Equations via a Novel Lucas Operational Matrix of Fractional Derivatives. Romanian Journal of Physics 61 (5-6), 795-813.

[C] Spectral Solutions for Multi-Term Fractional Initial Value Problems Using a New Fibonacci Operational Matrix of Fractional Integration. Progress in Fractional Differentiation and Applications 2 (2), 141-151.

[D] New Spectral Solutions of Multi-Term Fractional-Order Initial Value Problems With Error Analysis. CMES: Computer Modeling in Engineering & Sciences 105 (5), 375-398.

[E] New ultraspherical wavelets spectral solutions for fractional Riccati differential equations Abstract and Applied Analysis 2014.

There are same papers related to VPO systems in fractional order form.For example, Ge and Hsu numerically simulate various nonautonomous and autonomous fractional order generalized VPO systems [2]. They show that chaotic motions exist in the nonautonomous generalized VPO system excited by a sinusoidal time function. Ge and Zhang study anticontrol of chaos of modified VPO systems in fractional order form [4]. They found that chaos existed in the fractional order systems with order from 1.8 down to 0.6 for the addition of constant term, an efficient way to transform a non-chaotic dynamical system into a chaotic one. In the paper [5], chaotic behaviors of a fractional order modified van der Pol system are studied by phase portraits, Poincare´ maps and bifurcation diagrams. It is found that chaos exists in this system with order from 1.8 down to 0.8 much less than the number of states of the system.

In the paper [6], the influence of a fractional-order time derivative introduced in the Van der Pol equation dynamics and also forced version are investigated. The results show that fractional-order systems can exhibit different behaviour from those obtained with the standard VPO, depending on the order's derivative (or the system order).

In the article [7], a fractional order Bonhoeffer–van der Pol oscillator is used and its solutions of special type are analyzed. In particular, the results show that at different orders of fractional derivative indices the system under consideration can be more unstable than the regular one. The article [8] represents some familiar dynamic equations in fractal form including two different forms of fractional van der Pol equations and uses the method of harmonic balance to obtain approximate expressions for the transition curves in the fractional Mathieu equation:

In the paper [9], the effect of the fractional order of damping on the dynamic behaviors of the van der Pol equation is studied numerically. The forced, fractionally damped van der Pol equation was transformed into a set of fractional integral equations. An Adams–Bashforth–Moulton predictor–corrector method was used to solve the fractional integral equations.

In the study [10], the fractional van der Pol equation is solved by means of the variational iteration method. In the work presented in 2003 [11], it is calculated analytical approximations to periodic solutions of fractional van der Pol equations.

In this study, we use multiple time scale method and numerically by finite difference method to find solutions of VPO. There are some approximate solutions of the traditional van der Pol equation using multiple time scale method in the literature. One of these solutions is given in the textbook [12]. In the textbook, techniques of determining approximate solutions with some perturbation technique are described for nonlinear van der Pol equation. These analyses are limited to weakly nonlinear systems. In the paper [13], the MLP method, which is enable a strongly nonlinear system to be transformed into a small parameter system is presented to extend the range of application to strongly nonlinear systems. It is considered the bifurcations of a strongly nonlinear oscillator which is a more general Mathieu–van der Pol system. The asymptotic solution of fractional van der Pol oscillators is obtained by the two-scale method and verified the validity of the asymptotic solution using a numerical method [14].

In the paper [15], we present an alternative representation of the diffusion equation and the diffusion–advection equation using the fractional calculus approach, the spatial-time derivatives are approximated using the fractional definition recently introduced by Caputo and Fabrizio in the range $\beta,\gamma \in (0;2]$ for the space and time domain respectively.

We present an analysis based on a combination of the Laplace transform and homotopy methods in order to provide a new analytical approximated solutions of the fractional partial differential equations (FPDE) in the Liouville-Caputo and Caputo-Fabrizio sense [16].

The discrete fractional model is a fractionization of the classical difference equation and can be more suitable to depict the random or discrete phenomena compared with fractional partial differential equations [17].

In this paper [18]; we present the procedure to obtain analytical solutions of Liénard type model of a fluid transmission line represented by the Caputo-Fabrizio fractional operator. For such a model, we derive a new approximated analytical solution by using the Laplace homotopy analysis method.

An implicit scheme for the numerical approximation of the distributed order time-fractional reaction–diffusion equation with a nonlinear source term is presented [19].

It is difficult to find out the analytical solution of nonlinear fractional-order system. Therefore, in most of the literature in the area of fractional calculus, approximate and numerical solutions are important to analyze the nonlinear fractional-order equations.Hence, the multiple time scale method, approximate solution technique, and finite difference method, time-domain numerical method are used in this paper. The transient responses of fractional van der Pol equation are obtained by using the mentioned methods. The effects of fractional order on transient response are represented. The numerical finite difference method is used to verify the validity of the approximate solution.

2 The Approximate Solution by Multiple Time Scales (MTS) Method

In this study, the fractional van der Pol equation with a small fractional damping term is considered [8,11,14]

$$\ddot{x}(t) - \varepsilon(1 - x(t)^2)D^\alpha x(t) + x(t) \quad = 0 \qquad (\varepsilon << 1) \tag{1}$$

where x is the dynamical variable, t is independent variable, which denotes time in dynamical problems and ε is a positive small parameter. D^α is the fractional differential operator that denotes the αth derivative of the related function with respect to t and it is defined in the Caputo sense as [1,3,14]

$$D^\alpha x(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{x'(s)ds}{(t-s)^\alpha}, \; 0 < \alpha < 1 \tag{2}$$

$\Gamma(\cdot)$ is the gamma function. Initial conditions are

$$x(0) = a_0, \; \dot{x}(0) = a_1 \tag{3. a, b}$$

To determine approximate solution of Eq. (1) by using the method of multiple time scale, the desired function *x(t)* is represented in terms of series [12]

$$x(t) = x_o(T_0, T_1) + \varepsilon x_1(T_0, T_1) + \ldots \tag{4}$$

where T_n represent different time scales. The fast and slow time scales are

$$T_0 = t, \; T_1 = \varepsilon t \tag{5}$$

Then, the derivatives with respect t for first, second and fractional derivatives transform into [1 and 12]

$$\frac{d}{dt} = D_0 + \varepsilon D_1 + \ldots$$

$$\frac{d^2}{dt^2} = D_0^2 + 2\,\varepsilon D_0 D_1 + \ldots \tag{6. a, b, c}$$

$$(\frac{d}{dt})^\alpha = D_0^\alpha + \varepsilon\alpha D_0^{\alpha-1} D_1 + \ldots$$

where $D_n = \frac{\partial}{\partial T_n}$. Substituting Eqs. (4)-(6) into the differential equation (1) and boundary conditions (3), separating terms at each order of ε, the equations and boundary conditions at each order are obtained

$$O(1) \quad D_0^2 x_0 + x_0 = 0 \qquad (7.\text{ a, b})$$
$$x_0(0) = a_0,\ D_0 x_0(0) = a_1$$

$$O(\varepsilon) \quad D_0^2 x_1 + x_1 = -2D_0 D_1 x_0 + D_0{}^{\alpha} x_0 - x_0{}^2 D_0{}^{\alpha} x_0 \qquad (8.\text{a, b})$$
$$x_1(0) = 0,\ D_0 x_1(0) = 0$$

The solution at the first order is

$$x_0 = A(T_1)e^{iT_0} + \overline{A}(T_1)e^{-iT_0} \qquad (9)$$

where A and $\overline{A}$ are complex amplitudes and their conjugates, respectively. To determine structure of A and its complex conjugate, the relation (8) and solvability conditions are used. $A(T_1)$ is represented the complex amplitudes in polar form

$$A = \frac{1}{2}a(T_1)e^{i\beta(T_1)}$$

Eq. (9) is substituted into (8) and below relation can be obtained to eliminate secular terms

$$-2iD_1 A + Ai^{\alpha} - A^2\overline{A}(-i)^{\alpha} - 2A^2\overline{A}i^{\alpha} = 0 \qquad (10)$$

where

$$i^{\alpha} = \cos(\frac{\alpha\pi}{2}) + i\sin(\frac{\alpha\pi}{2}) = e^{\frac{\alpha\pi}{2}i} \qquad (11)$$

Substituting relationship (11) in Eq. (10) and separating the real and the imaginary part of the equation, below relations are obtained

$$-\frac{da}{dT_1} + \frac{a}{2}\sin(\frac{\alpha\pi}{2}) - \frac{a^3}{8}\sin(\frac{\alpha\pi}{2}) = 0 \qquad (12)$$

$$a\frac{d\beta}{dT_1} + \frac{a}{2}\cos(\frac{\alpha\pi}{2}) - \frac{3a^3}{8}\cos(\frac{\alpha\pi}{2}) = 0 \qquad (13)$$

The solutions of Eq. (12) and Eq. (13) are, respectively

$$a(T_1) = \left(\frac{4c_0 e^{T_1 \sin\left(\frac{\alpha\pi}{2}\right)}}{c_0 e^{T_1 \sin\left(\frac{\alpha\pi}{2}\right)} - 1}\right)^{\frac{1}{2}} \qquad (14)$$

$$\beta(T_1)=\frac{3}{2}\cot\left(\frac{\alpha\pi}{2}\right)\ln\left(c_0 e^{T_1\sin\left(\frac{\alpha\pi}{2}\right)}-1\right)-\frac{1}{2}T_1\cos\left(\frac{\alpha\pi}{2}\right)+c_1 \tag{15}$$

Applying initial conditions yields

$$c_0=\frac{a_1^2}{a_1^2-4} \quad c_1=-\frac{3}{2}\ln(c_0-1)\cot\left(\frac{\alpha\pi}{2}\right) \tag{16 a and b}$$

Therefore $A(T_1)$ is defined as follows

$$A(T_1)=\frac{1}{2}\left(\frac{4c_0 e^{T_1\sin\left(\frac{\alpha\pi}{2}\right)}}{c_0 e^{T_1\sin\left(\frac{\alpha\pi}{2}\right)}-1}\right)^{\frac{1}{2}}\exp\left(i\frac{3}{2}\cot\left(\frac{\alpha\pi}{2}\right)\ln\left(c_0 e^{T_1\sin\left(\frac{\alpha\pi}{2}\right)}-1\right)-\frac{1}{2}iT_1\cos\left(\frac{\alpha\pi}{2}\right)+ic_1\right) \tag{17}$$

The solution at the first order can be assumed to be approximate solution of the van der Pol equation. So, the approximate solution is

$$x\cong\left(\frac{4c_0 e^{\varepsilon t\sin\left(\frac{\alpha\pi}{2}\right)}}{c_0 e^{\varepsilon t\sin\left(\frac{\alpha\pi}{2}\right)}-1}\right)^{\frac{1}{2}}\cos\left(t+\frac{3}{2}\cot\left(\frac{\alpha\pi}{2}\right)\ln\left(c_0 e^{\varepsilon t\sin\left(\frac{\alpha\pi}{2}\right)}-1\right)-\frac{1}{2}\varepsilon t\cos\left(\frac{\alpha\pi}{2}\right)+c_1\right) \tag{18}$$

3 The Numerical Solution by Finite Difference Method (FDM) Method

Here a finite difference method is applied directly to the fractal van der Pol equation. Terms in Eq.(1) involving fractal derivatives are approximated with central difference:

$$\ddot{x}\cong\frac{x_{n+1}-2x_n+x_{n-1}}{\Delta t^2} \tag{19}$$

$$D^{\alpha}x\cong\frac{1}{\Gamma(1-\alpha)}\sum_{m=1}^{n}\frac{\Gamma(n-m-\alpha+1)}{(n-m)!}\left(\frac{x_m-x_{m-1}}{\Delta t^{\alpha}}\right) \tag{20}$$

In these discretized forms, a superscript indicates time step. Substituting these finite difference approximations into fractional van der Pol equation yields the discretized equation at the *n*th time step

$$x_{n+1}=2x_n-x_{n-1}+\varepsilon\left(1-x_n^{\,2}\right)\Delta t^2\frac{1}{\Gamma(1-\alpha)}\sum_{m=1}^{n}\frac{\Gamma(n-m-\alpha+1)}{(n-m)!}\frac{x_m-x_{m-1}}{\Delta t^{\alpha}}-\Delta t^2 x_n \tag{21}$$

where $n=1, 2, 3,\ldots$

For simplicity, backward difference with respect to time is used at the boundary condition:

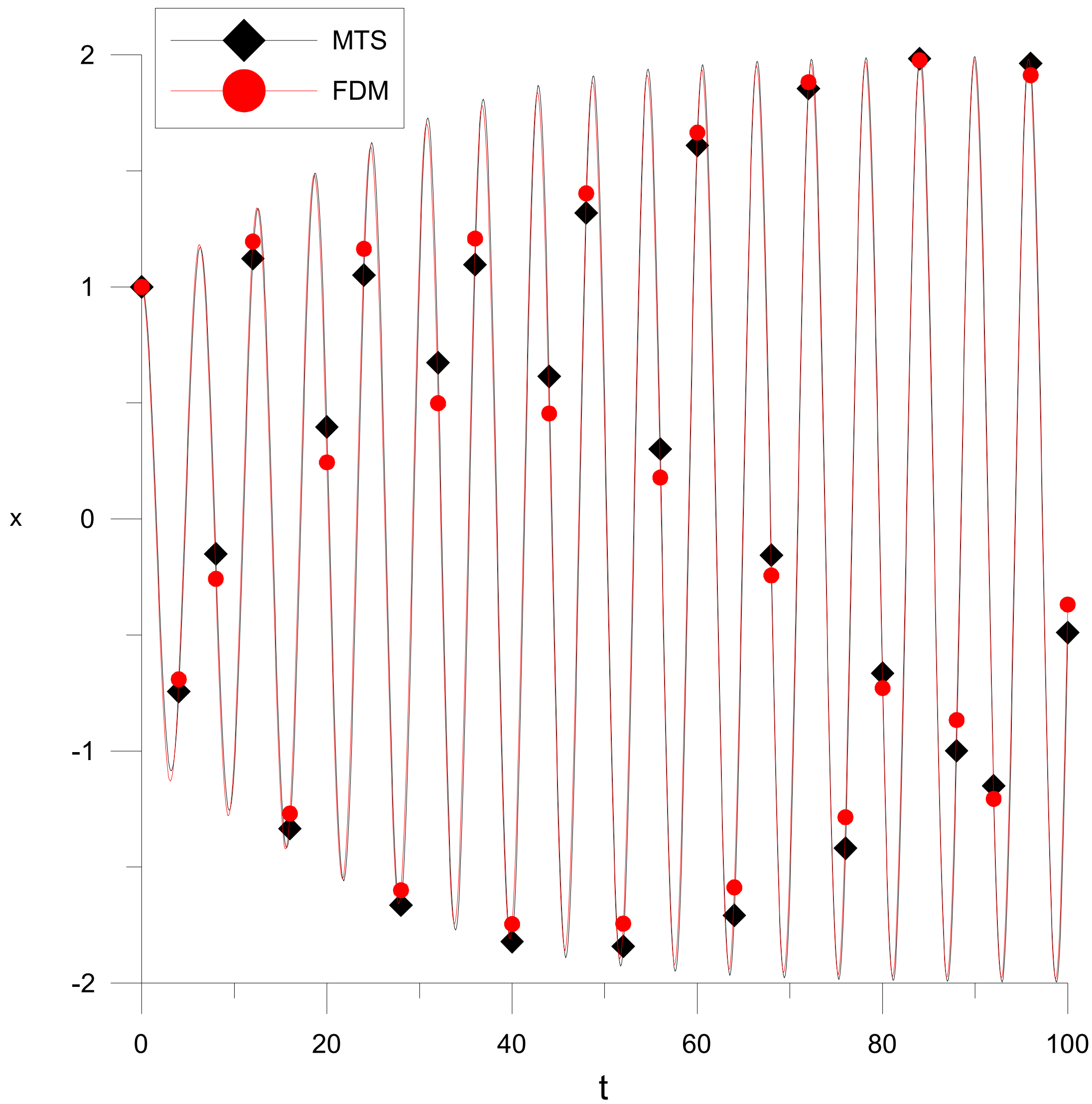
$$\dot{x}\cong\frac{x_n-x_{n-1}}{\Delta t} \tag{22}$$

And then, from initial conditions given relations (3 a and b):

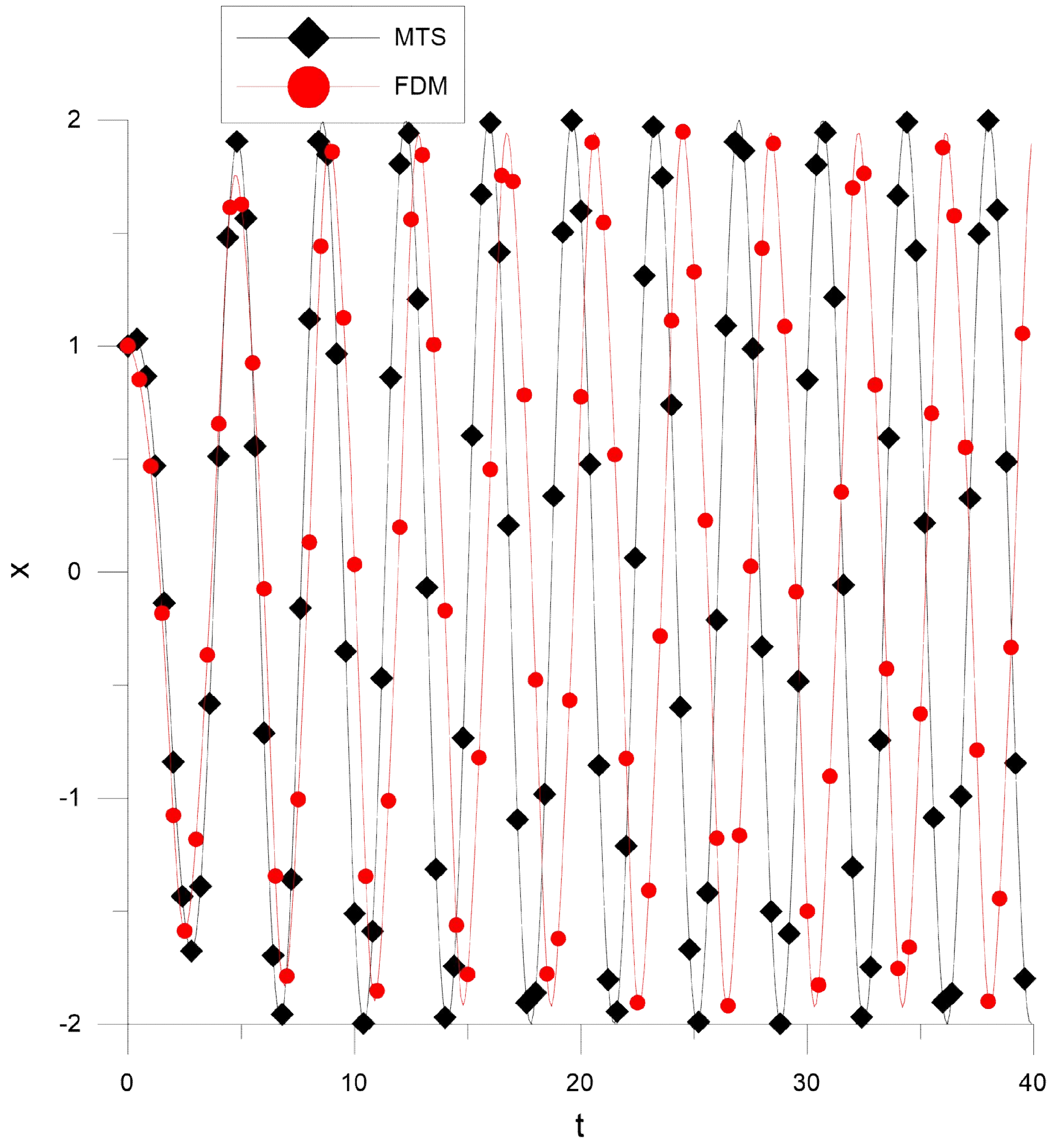
$$x_0 = a_0, \qquad x_{-1} = a_0 - \Delta t a_1 \tag{23}$$

4 Numerical Results

For simplicity, it is used that $a_0=1.0$ and $a_1=0.0$ in numerical results. The numerical solutions are compared with the analytical solution in Fig. 1. For small ε values, the analytical and numerical results are same. However, the analytical solution is inaccurate for great ε values, for example ε≥1.0. It means that the approximate solution is valid only for weakly nonlinear van der Pol equation. It is expected conclusion, because the multiple time scale method is developed to solve weakly nonlinear dynamic problems.



(a)



(b)

Fig. 1. When α=0.5, comparisons of multiple time scale (MTS) and finite difference method (FDM) (a) for ε=0.1, (b) for ε=1.0

In Fig. 2, the effect of fractal derivatives are represented. Initially, as t increases, the amplitude of vibration increases. Then, there is no variation on the amplitude after it has a certain value. The classical damping term which has integer derivatives has almost no effect of natural frequencies. However, the fractal derivative term has ability to change meaningfully the natural frequencies. The fractal derivative approach would be to combine the effects of stiffness and damping in a single term. The fractal coefficient, α, act on variation of the time, the amplitudes of vibrations reach the certain value. When α increase, the damping effect is increased and then the amplitude reaches constant values in a shorter term. However, the certain amplitude value does not change with α.

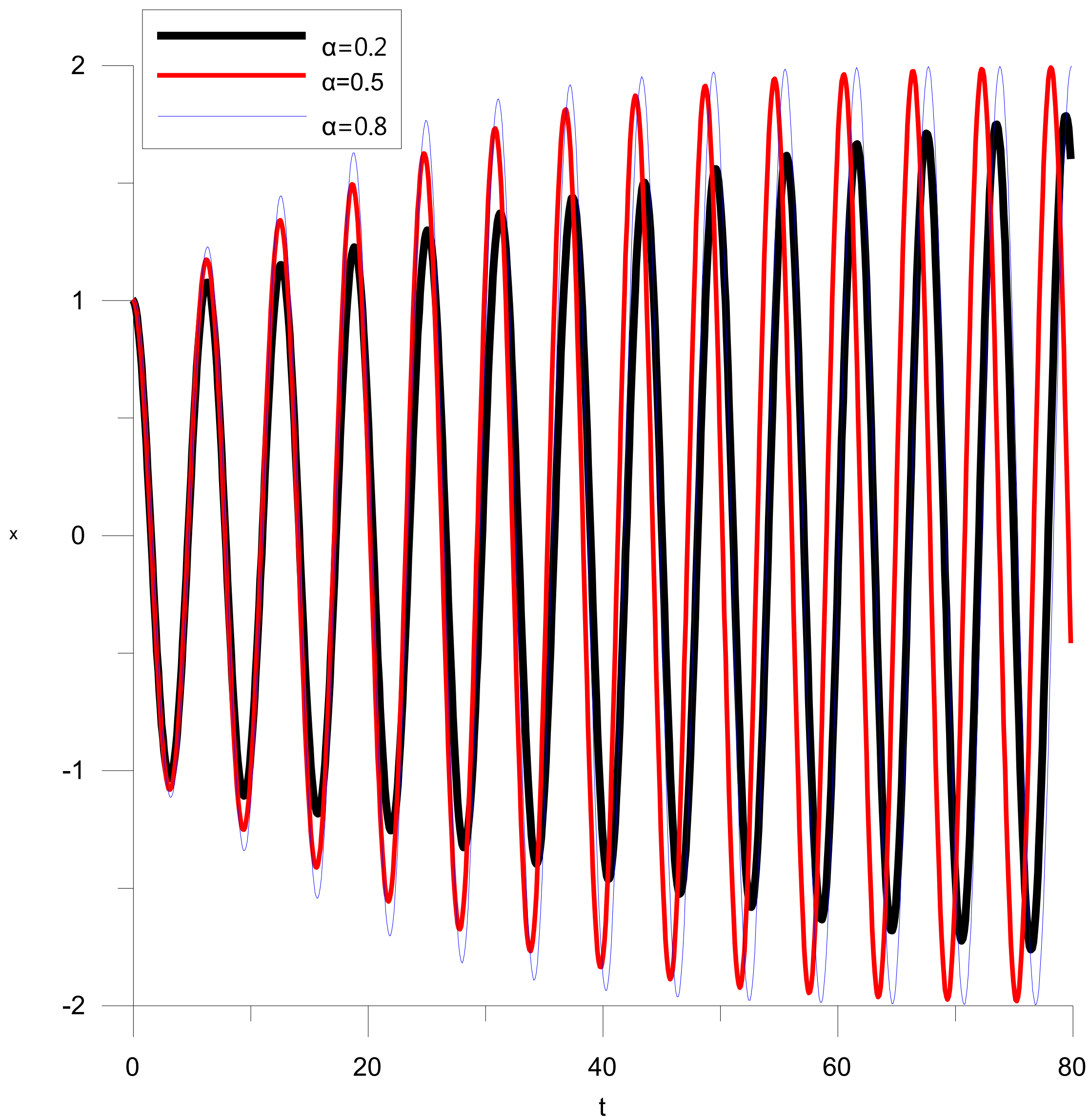


Fig. 2. Effect of fractal derivatives order α for ε=0.1

5 Conclusion

In this study, to show effect of fractal derivative on van der Pol oscillator, the approximate and numerical solutions are obtained. The approximate solution is compared with numerical solution. Then, the validity of the approximate solution is proved by numerical methods.

The weakly nonlinear van der Pol equation is considered and its solutions are determined by using the multiple scale method and finite difference method. The possible solutions of this oscillator are discussed for different values α and ε. The solution obtained by using multiple time scale method is accurate only for small parameter ε.

The fractional derivative order α has meaningful effect on natural frequencies of the oscillator. It is expected consequences because the fractal derivatives can be used instead of both spring and damping terms. However, in the real life, the model including fractional damping term can provide better results than including conventional damping term.

Competing Interests

Author has declared that no competing interests exist.

References

[1] Rossikhin Yuriy A, Marina Shitikova V. Application of fractional calculus for dynamic problems of solid mechanics: Novel trends and recent results. Applied Mechanics Reviews. 2010;63(010801):1-52.

[2] Zheng-Ming Ge, Mao-Yuan Hsu. Chaos in a generalized van der Pol system and in its fractional order system. Chaos, Solutions and Fractals. 2007;33:1711–1745.

[3] Shantanu Das. Functional fractional calculus for system identification and controls. Springer-Verlag Berlin; 2008.

[4] Zheng-Ming Ge, An-Ray Zhang. Anti control of chaos of the fractional order modified van der Pol systems. Applied Mathematics and Computation. 2007;187:1161–1172.

[5] Zheng-Ming Ge, An-Ray Zhang. Chaos in a modified van der Pol system and in its fractional order systems. Chaos, Solitons and Fractals. 2007;32:1791–1822.

[6] Ramiro S. Barbosa JA. Tenreiro Machado Vinagre BM, Calderón AJ. Analysis of the Van der Pol oscillator containing derivatives of fractional order. Journal of Vibration and Control. 2007;13:1291.

[7] Gafiychuka V, Datskoc B, Meleshkoc V. Analysis of fractional order Bonhoeffer–van der Pol oscillator. Physica A. 2008;387:418–424.

[8] Rand RH, Sah SM, Suchorsky MK. Fractional Mathieu equation. Commun Nonlinear Sci Numer Simulat. 2010;15:3254–3262.

[9] Chen JH, Chen WC. Chaotic dynamics of the fractionally damped van der Pol equation. Chaos, Solitons and Fractals. 2008;35:188–198.

[10] A Konuralp et al. Numerical solution to the van der Pol equation with fractional damping. Phys. Scr. 2009;T136:014034.

[11] Mickens RE. Fractional van der Pol Equations. Journal of Sound and Vibration. 2003;259(2):457–460.

[12] Nayfeh AH. Introduction to perturbation techniques. A Wiley-Interscience Publication; 1980.

[13] Jiashi Tang, Jinqi Qin, Han Xiao. Bifurcations of a generalized van der Pol oscillator with strong nonlinearity. Journal of Sound and Vibration. 2007;306(2007):890–896.

[14] Feng Xie, Xueyuan Lin. Asymptotic solution of the van der Pol oscillator with small fractional damping. Phys. Scr. 2009;T136(014033):4.

[15] Gómez-Aguilar JF, López-López MG, Alvarado-Martínez VM, Reyes-Reyes J, Adam-Medina M. Modeling diffusive transport with a fractional derivative without singular kernel. Physica A: Statistical Mechanics and its Applications. 2016;447:467-481.

[16] Morales-Delgado VF, Gómez-Aguilar JF, Yépez-Martínez H, Baleanu D, Escobar-Jimenez RF, Olivares-Peregrino VH. Laplace homotopy analysis method for solving linear partial differential equations using a fractional derivative with and without kernel singular. Advances in Difference Equations. 2016(1):1.

[17] Wu GC, Baleanu D, Zeng SD, Deng. Discrete fractional diffusion equation. Nonlinear Dynamics. 2015;80(1-2):281-286.

[18] Gómez-Aguilar JF, Torres L, Yépez-Martínez H, Baleanu D, Reyes JM, Sosa IO. Fractional Liénard type model of a pipeline within the fractional derivative without singular kernel. Advances in Difference Equations. 2016(1):1-13.

[19] Morgado ML, Rebelo M. Numerical approximation of distributed order reaction–diffusion equations. Journal of Computational and Applied Mathematics. Z.G. 2015;275:216-227.

Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)
http://sciencedomain.org/review-history/17597