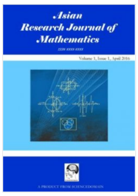

Asian Research Journal of Mathematics
2(3): 1-11, 2017; Article no.ARJOM.31092
ISSN: 2456-477X

SCIENCEDOMAIN *international*
www.sciencedomain.org

Common Caristi-type Fixed Point Theorem for Two Single Valued Mappings in Cone Metric Spaces

Lazaiz Samih[1], Aamri Mohamed[1], Saffaj Hamza[1] and Zakary Omar[2*]

[1] *Laboratory of Algebra Analysis and Applications, Department of Mathematics and Computer Science, Faculty of Sciences Ben M'Sik, Hassan II University of Casablanca, BP 7955, Sidi Othman, Casablanca, Morocco.*
[2] *Laboratory of Analysis Modeling and Simulation, Department of Mathematics and Computer Science, Faculty of Sciences Ben M'Sik, Hassan II University of Casablanca, BP 7955, Sidi Othman, Casablanca, Morocco.*

Authors' contributions

All authors contributed equally and significantly to writing this paper. All authors read and approved the final manuscript.

Article Information

DOI: 10.9734/ARJOM/2017/31092
Editor(s):
(1) Dijana Mosic, Department of Mathematics, University of Nis, Serbia.
Reviewers:
(1) Lakshmi Narayan Mishra, Mody University of Science and Technology, Lakshmangarh, Sikar Road, Sikar, Rajasthan 332 311, India.
(2) K. Prudhvi, Osmania University, India.
(3) Choonkil Park, Hanyang University, Republic of Korea.
(4) Zaid Mohammed Fadail, Universiti Kebangsaan Malaysia, Malaysia.
(5) Nihal Tas, Balikesir University, Turkey.
Complete Peer review History: http://www.sciencedomain.org/review-history/17663

Received: 19th December 2016
Accepted: 19th January 2017
Published: 29th January 2017

Original Research Article

Abstract

In this paper, we obtain a common Caristi-type fixed point theorem for two single valued mappings in the setting of cone metric spaces. Further, we derive some consequences and a coupled fixed point theorem for two mappings without the need of the monotonicity assumption. Our work is supported by different examples.

Keywords: Caristi; Fixed point; Common fixed point; Coupled fixed point; Cone metric space.

2010 Mathematics Subject Classification: 47H10, 54H25.

**Corresponding author: E-mail: zakaryma@gmail.com*

1 Introduction

In 1972, the celebrated variational principle of Ekeland [1] for approximate solutions of non-convex minimization problems has appeared for the first time. It has received a great deal of attention and has been applied to numerous problems in several fields [2, 3]. It is a useful tool to solve problems in optimization, optimal control theory, game theory, nonlinear equations and dynamical systems (see [4, 5, 2, 3, 6]). There have appeared subsequently many extended and generalized versions of that principle as seen in the references [7, 8, 9, 10, 11].

In [12, 13], the mathematician Caristi proved a result which is one of the most important generalization of Banach principle [14] for maps of a complete metric space into itself. And it is a variation and equivalent to the well known ε-variational principle of Ekeland [1, 2, 3]. In the literature, there have appeared also many extensions and equivalent formulations of Caristi's fixed point theorem (see [15, 16, 17, 18, 19, 20] and references therein).

The study of common fixed points of mappings satisfying certain contractive conditions is one of the main concerns of researchers over the last few decades [21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31]. In 1976, Jungck [32], proved a common fixed point theorem for commuting maps, generalizing the Banach contraction principle. Although, these results require the continuity of one of the two maps involved. Sessa [33] introduced the notion of weakly commuting maps. Jungck [34] invented the term compatible mappings in order to generalize the concept of weak commutativity and showed that weakly commuting maps are compatible but the converse is not true. Pant [35] defined R-weakly commuting maps and proved common fixed point theorems, assuming also the continuity of at least one of the mapping. While Kannan [36] proved the existence of a fixed point for a map that can have a discontinuity in a domain, however the maps involved in every case were continuous at the fixed point. Jungck [37, 38] defined a pair of self mappings to be weakly compatible if they commute at their coincidence points.

In 2007, Huang and Zhang [39] introduced the notion of cone metric spaces as a generalization of metric spaces. They introduced the concept of convergence in cone metric spaces via the interior of the cone and obtained some fixed point theorems for contractive mappings. Authors in [40, 41, 39, 42, 43] obtained fixed point theorems on cone metric spaces under assumption that the cone is normal. Also, the authors in [44, 45, 46] proved fixed point results under assumption that the cone is regular.

In [47], Cho and Bae gave an extension of Caristi's theorem in the setting of complete cone metric spaces, and they proved that this result and Ekeland variational principle are equivalent.

Following this direction, in this paper, we obtain a common Caristi-type fixed point theorem for two single valued mappings in the setting of cone metric spaces. Further, we derive some consequences and a coupled fixed point theorem for two mappings without the need of the monotonicity assumption. We give some examples to support our work.

2 Preliminaries

For the convenience of the reader we repeat the relevant material from [47] without proofs, thus making our exposition self-contained.

Let E be a topological vector space. A subset $P \subset E$ is called a convex cone if :

1. $P + P \subset P$
2. for every $\lambda > 0$, $\lambda P \subset P$
3. $P \cap (-P) = \{\theta\}$, where θ denotes the zero of E.

It is well-known that a convex cone $P \subset E$ generates a partial-ordering on E (i.e. a reflexive, anti-symmetric and transitive relation) by

$$x \preceq y \Leftrightarrow y - x \in P.$$

Definition 2.1. A cone P is strongly minihedral if every upper bounded nonempty subset A of E, $\sup A$ exists in E. Dually, a cone P is strongly minihedral if every lower bounded nonempty subset A of E, $\inf A$ exists in E.

Definition 2.2. A strongly minihedral cone P is continuous if, for any bounded chain $(x_\alpha)_{\alpha\in\Gamma}$ we have

$$\inf_{\alpha\in\Gamma} \|x_\alpha - \inf\{x_\alpha;\, \alpha\in\Gamma\}\| = 0$$

and

$$\sup_{\alpha\in\Gamma} \|x_\alpha - \sup\{x_\alpha;\, \alpha\in\Gamma\}\| = 0.$$

Throughout this paper, assume that $(E, \|.\|)$ is normed vector space and P is a convex cone in E.

Definition 2.3. Let X be a nonempty set and $\delta : X \times X \to E$ a mapping satisfying for all $x, y, z \in X$:

(i) $\theta \preceq \delta(x, y)$ and $\delta(x, y) = \theta$ if and only if $x = y$,

(ii) $\delta(x, y) = \delta(y, x)$,

(iii) $\delta(x, z) \preceq \delta(x, y) + \delta(y, z)$.

Then δ is called a cone metric on X and (X, δ) is called a cone metric space.

For $x, y \in E$, $x \ll y$ stand for $y - x \in \operatorname{int}(P)$, where $\operatorname{int}(P)$ is the interior of P.
In this paper, we use the concept of regularity to obtain our results.

Definition 2.4.

1. A sequence $(x_n)_n$ of a cone metric space (X, δ) converges to a point $x \in X$ if for any $c \in \operatorname{int}(P)$, there exists $N \in \mathbb{N}$ such that for all $n > N$, $\delta(x_n, x) \ll c$. Denoted by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$.
2. A sequence $(x_n)_n$ of a cone metric space (X, δ) is Cauchy if for any $c \in \operatorname{int}(P)$, there exists $N \in \mathbb{N}$ such that for all $n, m > N$, $\delta(x_n, x_m) \ll c$.
3. A cone metric space (X, δ) is called complete if every Cauchy sequence is convergent.

Lemma 2.1. *Let (X, δ) be a cone metric space over a cone P in E. Then one has the following.*

1. *$Int(P) + Int(P) \subset Int(P)$ and $\lambda Int(P) \subset Int(P)$, $\lambda > 0$.*
2. *If $c \gg \theta$, then there exists $\gamma > 0$ such that $\|b\| < \gamma$ implies $b \ll c$.*
3. *For any given $c \gg \theta$ and $c_0 \gg \theta$ there exists $n_0 \in \mathbb{N}$ such that $\frac{c_0}{n_0} \ll c$.*
4. *If (a_n) , (b_n) are sequences in E such that $a_n \to a$, $b_n \to b$ and $a_n \preceq b_n$ for all $n \geq 0$, then $a \preceq b$.*

Definition 2.5. A function $\varphi : X \longrightarrow E$ is called lower semi-continuous if, for every sequence $(x_n)_n \subset X$ converging to some point $x \in X$ and satisfying $\varphi(x_{n+1}) \preceq \varphi(x_n)$ for all $n \in \mathbb{N}$, we have $\varphi(x) \preceq \liminf \varphi(x_n) := \sup_{n\in\mathbb{N}} \inf_{m\geq n} \varphi(x_m)$.

Definition 2.6. Let (X, δ) be a cone metric space. A mapping $T : X \longrightarrow X$ is sequentially continuous if for each sequence (x_n) which converge to $x \in X$ we have $(Tx_n)_n$ is convergent in X and $\lim_{n\to\infty} Tx_n = Tx$.

In [47], Cho and Bae gave an extension of Caristi's theorem in the setting of a complete cone metric space over a strongly minihedral and continuous cone. In the sequel we will need the following result.

Theorem 2.2 (Cho-Bae [47]). *Let (X, δ) be a complete cone metric space such that P is strongly minihedral and continuous. And, let $T : X \longrightarrow X$ be a mapping satisfying for each x in X*

$$\delta(x, Tx) \preceq \varphi(x) - \varphi(Tx), \tag{2.1}$$

where $\varphi : X \longrightarrow P$ is lower semi continuous, then T has a fixed point.

3 Main Results

Theorem 3.1. *Let (X, δ_i) be a complete cone metric space $(i = 1, 2)$ such that P is strongly minihedral and continuous cone. Let $T, S : X \longrightarrow X$ be two mappings, and $f, g : X \longrightarrow \mathbb{R}^+$ be two functions such that for some $\varepsilon > 0$*

$$\begin{cases} \sup\{f(x) \mid x \in X,\ \varphi(x) \leqslant \inf_{z\in X} \varphi(z) + \varepsilon\} < \infty \\ \sup\{g(x) \mid x \in X,\ \varphi(x) \leqslant \inf_{z\in X} \varphi(z) + \varepsilon\} < \infty \end{cases}, \tag{3.1}$$

where $\varphi : X \longrightarrow P$ is lower semi continuous.

Suppose that for each $(x, y) \in X^2$ we have

$$\begin{cases} \delta_1(x, Tx) \preceq f(x)(\varphi(x) - \varphi(Sy)) \\ \delta_2(y, Sy) \preceq g(y)(\varphi(y) - \varphi(Tx)) \end{cases}, \tag{3.2}$$

then there exists $\bar{x} \in X$ such that $\bar{x} = T\bar{x} = S\bar{x}$.

Proof. Let $\varepsilon > 0$ and put

$$X_1 = \left\{x \in X \mid \varphi(x) \preceq \inf_{z\in X} \varphi(z) + \varepsilon\right\}$$

and

$$\alpha = \max\left\{\sup_{z\in X_1} f(z),\ \sup_{z\in X_1} g(z)\right\} < \infty$$

We note that X_1 is nonempty set and since φ is lower semi continuous function, then X_1 is a closed subset of X, so X_1 is a complete subset.

Let $\psi(x, y) = \alpha(\varphi(x) + \varphi(y))$ and $\rho((x, y), (z, t)) = \delta_1(x, z) + \delta_2(y, t)$ for all x, y, z, t in X_1, using the inequalities (3.2) we define a single valued mapping $L : X_1 \times X_1 \longrightarrow X \times X$ by $L_1(x, y) = (Tx, Sy)$ such that

$$\rho((x, y), (u, v)) \preceq \psi(x, y) - \psi(u, v). \tag{3.3}$$

Let $X_2 = \left\{(x, y) \in X_1^2 \mid \psi(x, y) \preceq \inf_{(z,t)\in X_1^2} \psi(z, t) + \varepsilon\right\}$, the same reasoning applied to (X_2, ρ) shows that is non-empty complete subset of X^2 (note that ψ is also lower semi continuous) and it is stable by the mapping $L(x, y) = (Tx, Sy)$ i.e. $L(X_2) \subseteq X_2$, indeed for all $(x, y) \in X_2$ using the inequality (3.3) we get

$$\psi(L(x, y)) \preceq \psi(x, y) \preceq \inf_{(z,t)\in X_1^2} \psi(z, t) + \varepsilon$$

and thus $L(x,y) \in X_2$, so L is a self map of X_2.

By Theorem 2.2, there exists $(\overline{x},\overline{y}) \in X_2$ such that

$$L(\bar{x},\bar{y}) = (\bar{x},\bar{y}) \Leftrightarrow T\bar{x} = \bar{x} \text{ and } S\bar{y} = \bar{y}$$

We get by the second inequality of (3.2)

$$\delta_2(\bar{x}, S\bar{x}) \preceq \alpha(\varphi(\bar{x}) - \varphi(T\bar{x})) = 0,$$

which completes the proof. □

Theorem 3.2. *Let (X,δ) be a complete cone metric space such that P is strongly minihedral and continuous, T and S two self mappings of X. If there exist functions φ and ψ from X into P such that for all x in X*

$$\begin{cases} \delta(x,Tx) \preceq \varphi(x) - \varphi(STx) \\ \delta(x,Sx) \preceq \psi(x) - \psi(TSx) \end{cases} \tag{3.4}$$

where $\varphi \circ S$ and ψ are lower semi continuous, then T and S admit at least one common fixed point.

Proof. The first inequality of (3.4) implies that

$$\delta(Sx, TSx) \preceq \varphi(Sx) - \varphi(STSx)$$

for all $x \in X$, then

$$\begin{aligned}\delta(x, TSx) &\preceq \delta(x,Sx) + \delta(Sx,TSx) \\ &\preceq \psi(x) - \psi(TSx) + \varphi(Sx) - \varphi(STSx).\end{aligned}$$

We put $\phi(x) = \psi(x) + \varphi(Sx)$, hence

$$\delta(x,TSx) \preceq \phi(x) - \phi(TSx),$$

so by Theorem 2.2 the mapping TS has a fixed point, that is, there exists $\bar{x}$ such that $TS\bar{x} = \bar{x}$. Using the second inequality of (3.4) we get

$$\delta(\bar{x}, S\bar{x}) \preceq \psi(\bar{x}) - \psi(TS\bar{x}) = 0$$

which implies that $S\bar{x} = \bar{x}$ and since $TS\bar{x} = \bar{x}$ we have $T\bar{x} = \bar{x}$, the proof is completed. □

Example 3.1. *Let $X = \left[0, \frac{1}{2}\right]$ endowed by the following cone-distance*

$$\delta(x,y) = \left(|x-y|, \frac{|x|+|y|}{2}\right)$$

and the cone $P = \left\{(x,y) \in \mathbb{R}^2 / x \geq 0,\ y \geq 0\right\}$. Let T, S, φ and ψ be as follows :

$$Tx = x^2, \quad Sx = x^3$$

and

$$\varphi(x) = \left(\sqrt[6]{x} - x, \sqrt[6]{x} - x\right), \ \ \psi(x) = 2\varphi(x).$$

It is clear that $\varphi \circ S$ is lower semi continuous.

Note that for each $x \in \left[0, \frac{1}{2}\right]$ we get

$$\begin{cases} x^5(2-x)^6 \leq 1 \\ x^5(3+x)^6 \leq 2^6 \end{cases}$$

which implies that

$$\begin{cases} x - x^2 \leq \sqrt[6]{x} - x \\ \frac{x+x^2}{2} \leq \sqrt[6]{x} - x \end{cases} \quad \Leftrightarrow \ \delta\left(x, Tx\right) \preceq \varphi\left(x\right) - \varphi\left(STx\right)$$

in the same manner we obtain

$$\begin{cases} x - x^3 \leq 2\left(\sqrt[6]{x} - x\right) \\ \frac{x+x^3}{2} \leq 2\left(\sqrt[6]{x} - x\right) \end{cases} \quad \Leftrightarrow \ \delta\left(x, Tx\right) \preceq \psi\left(x\right) - \psi\left(STx\right).$$

then for all $x \in \left[0, \dfrac{1}{2}\right]$ we have

$$\begin{cases} \delta(x, Tx) & \preceq & \varphi(x) - \varphi(STx) \\ \delta(x, Sx) & \preceq & \psi(x) - \psi(TSx) \end{cases}$$

thus, all assumptions of Theorem 3.2 are satisfied and $T0 = S0 = 0$.

Corollary 3.1. *Under the assumptions of Theorem 3.2 with $\rho : X \times X \longrightarrow P$ is a mapping satisfies for all $(x, y) \in X^2$ $\rho\left(x, y\right) = \theta \Rightarrow x = y$ and, for all $x \in X$*

$$\begin{cases} \delta(x, Tx) \preceq \varphi(Sx) - \varphi(STx) \\ \rho(x, Sx) \preceq \varphi(x) - \varphi(Tx) \end{cases} \tag{3.5}$$

then T and S admit a common fixed point.

Proof. Put $\phi\left(x\right) = \varphi(x) + \varphi(Sx)$ for all x in X, by (3.5) we have

$$\delta(x, Tx) \preceq \phi(x) - \phi(Tx)$$

Theorem 2.2 shows that T has a fixed point $\bar{x}$ in X, it follows from the second inequality of (3.5) that $S\bar{x} = \bar{x}$, which completes the proof. □

Corollary 3.2. *Let (X, δ) be a complete cone metric space such that P is strongly minihedral and continuous, T and S are two self mappings of X and $\rho : X \times X \longrightarrow P$ a mapping satisfying for all $(x, y) \in X^2$ $\rho\left(x, y\right) = \theta \Rightarrow x = y$.*

If there exist two mappings φ and ψ from X into P with only φ is lower semi continuous such that for all $x \in X$

$$\begin{cases} \delta(x, Tx) \preceq \varphi(x) - \psi(Sx) \\ \rho(x, Sx) \preceq \psi(Sx) - \varphi(Tx) \end{cases} \tag{3.6}$$

Then T and S have a common fixed point in X.

Proof. By assumptions on φ and ψ we get for all $x \in X$:

$$\varphi\left(Tx\right) \preceq \psi\left(Sx\right) \preceq \varphi\left(x\right)$$

and by the first inequality of (3.5) we have $\delta(x, Tx) \preceq \varphi(x) - \varphi(Tx)$ for all $x \in X$, then Theorem 2.2 states that T has at least one fixed point in X, set $T\bar{x} = \bar{x}$ thus $S\bar{x} = \bar{x}$; indeed

$$\begin{aligned} \rho(\bar{x}, S\bar{x}) &\preceq \psi(S\bar{x}) - \varphi(T\bar{x}) \\ &= \psi(S\bar{x}) - \varphi(\bar{x}) \\ &\preceq \psi(S\bar{x}) - \psi(\bar{Sx}) \\ &\preceq \theta, \end{aligned}$$

the proof is completed. □

Theorem 3.3. *Let (X, δ) be a complete cone metric space such that P is strongly minihedral and continuous, ρ a mapping of $X \times X$ into P such that for all $(x, y) \in X^2$: $\rho(x, y) = \theta \Rightarrow x = y$ and T, S two sequentially continuous self mappings of X, if there exist two functions φ and ψ from X to P such that for all $x \in X$*

$$\begin{cases} \delta(x, Tx) \preceq \varphi(Sx) - \varphi(STx) \\ \rho(x, Sx) \preceq \psi(x) - \psi(Tx) \end{cases} \tag{3.7}$$

then T and S admit at least one fixed point.

Proof. Let $x \in X$ and define a sequence $\{x_n\}_n$ by $x_{n+1} = T^n x$. By the first inequality of (3.7) we get for all $n \in \mathbb{N}$:

$$\delta(x_n, x_{n+1}) \preceq \varphi(Sx_n) - \varphi(Sx_{n+1})$$

i.e. $\{x_n\}_n$ is a Cauchy sequence, since X is a complete cone metric space, $\{x_n\}$ converges to some $\bar{x}$ in X. Note that T and S are sequentially continuous i.e. $\lim Tx_n = \lim Sx_n = t$ for some $t \in X$ then $\lim STx_n = Tt$ and $\lim TSx_n = St$, since $\lim Tx_n = \lim x_n = \bar{x} = t$ we get

$$\lim STx_n = \lim Sx_n = T\bar{x} = \bar{x}$$

so by the second inequality of (3.7), $S\bar{x} = \bar{x}$, which completes the proof. □

We give an application of Theorem 3.1 in the setting of coupled fixed point theorems without monotonicity.

Definition 3.1. An element $(x, y) \in X \times X$ is called a coupled fixed point of the mapping $F : X \times X \longrightarrow X$ if $x = F(x, y)$ and $y = F(y, x)$.

Theorem 3.4. *Let (X, δ_i) be a complete cone metric space ($i = 1, 2$) such that P is strongly minihedral and continuous, $F, G : X \times X \longrightarrow X$ two single valued mappings. If there exists a lower semi continuous function $\varphi : X \longrightarrow P$ such that*

$$\begin{cases} \delta_1(x, F(x, y)) \preceq \varphi(x) - \varphi(G(x, y)) \\ \delta_2(y, G(x, y)) \preceq \varphi(y) - \varphi(F(x, y)) \end{cases} \tag{3.8}$$

for each $(x, y) \in X \times X$. Then there exists $(\overline{x}, \overline{y}) \in X \times X$ such that $\overline{x} = F(\overline{x}, \overline{y})$ and $\overline{y} = G(\overline{x}, \overline{y})$.

Proof. We define a mapping $L : X \times X \longrightarrow X \times X$ by $L(x, y) = (F(x, y), G(x, y))$ and let $\psi(x, y) = \varphi(x) + \varphi(y)$ and $\rho((x, y), (z, t)) = \delta_1(x, z) + \delta_2(y, t)$ for all x, y, z, t in X. Using (3.8) we get for each $(x, y) \in X^2$

$$\rho((x, y), L(x, y)) \preceq \psi(x, y) - \psi(L(x, y))$$

by Theorem 2.2, there exists $(\overline{x}, \overline{y}) \in X \times X$ such that

$$(\overline{x}, \overline{y}) = L(\overline{x}, \overline{y}) = (F(\overline{x}, \overline{y}), G(\overline{x}, \overline{y}))$$

that is

$$\overline{x} = F(\overline{x}, \overline{y})$$
$$\overline{y} = G(\overline{x}, \overline{y})$$

□

If we drop the condition that φ is lower semi continuous, and replace it by the mapping $x \longmapsto \delta(Sx, Tx)$ is lower semi continuous we get the following result.

Theorem 3.5. *Let (X, δ) be a complete cone metric space such that P is strongly minihedral and continuous, T and S two self mappings of X with $TX \subseteq SX$ and $\varphi : X \longrightarrow P$ an arbitrary mapping. If $x \longmapsto \delta(Sx, Tx)$ is lower semi continuous such that for all x in X*

$$\max\{\delta(x, Tx), \delta(x, Sx)\} \preceq \varphi(Sx) - \varphi(Tx) \tag{3.9}$$

then T and S have a common fixed point.

Proof. Let x_0 be an arbitrary element of X, we define a sequence $\{y_n\}_n$as follows: since $TX \subseteq SX$ there exists $x_1 \in X$ such that

$$Tx_0 = Sx_1 = y_0$$

then there exists $x_2 \in X$ such that

$$Tx_1 = Sx_2 = y_1$$

so on until define y_n by induction : $Tx_n = Sx_{n+1} = y_n$. Set $\psi = 2\varphi$, and by (3.9) we get for all n in $\mathbb{N}^\star$

$$\begin{aligned}\delta(Sx_n, Tx_n) &\preceq \delta(x_n, Tx_n) + \delta(x_n, Sx_n)\\ &\preceq \psi(Sx_n) - \psi(Tx_n)\end{aligned}$$

which implies that $\delta(y_{n-1}, y_n) \preceq \psi(y_{n-1}) - \psi(y_n)$ so $\{y_n\}_n$ is a Cauchy sequence, then converges to $\overline{x} \in X$, hence

$$\lim Tx_n = \lim Sx_n = \lim y_n = \bar{x}$$

and

$$\begin{aligned}\delta(S\bar{x}, T\bar{x}) &\preceq \liminf \delta(Sx_n, Tx_n)\\ &\preceq \liminf (\psi(Sx_n) - \psi(Tx_n)) = \theta\end{aligned}$$

thus $S\bar{x} = T\bar{x}$, and by (3.9) $S\bar{x} = T\bar{x} = \bar{x}$. □

Example 3.2. *Let $X = L^\infty[0, 1]$, and let $E = \mathbb{R}^2$ and $P = \{(x, y)|x, y \geq 0\}$. We define $\delta : X \times X \longrightarrow P$ by*

$$\delta(h, k) = \left(\|h - k\|_\infty, \|h - k\|_1\right)$$

and take $\delta_1 = \delta_2 = \delta$. Then (X, δ) is a complete cone metric space, and P is strongly minihedral and continuous (see [48]).

We define $T : X \longrightarrow X$ and $S : X \longrightarrow X$ by $Th = \frac{1}{2}h$ and $Sk = \frac{3}{4}k$. Since T and S are continuous and so δ we have $x \longmapsto \delta(Sx, Tx)$ is lower semi continuous.

And we define a mapping $\varphi :\ X \longrightarrow P$ by

$$\varphi(h) = 2\left(\|h\|_\infty, \|h\|_1\right)$$

For any $h \in X$ it is clear that

$$\max\left\{\left(\frac{1}{2}\|h\|_\infty, \frac{1}{2}\|h\|_1\right), \left(\frac{1}{4}\|h\|_\infty, \frac{1}{4}\|h\|_1\right)\right\} = \left(\frac{1}{2}\|h\|_\infty, \frac{1}{2}\|h\|_1\right)$$

and

$$\varphi(Sh) - \varphi(Th) = 2\left(\frac{1}{4}\|h\|_\infty, \frac{1}{4}\|h\|_1\right) = \frac{1}{2}\left(\|h\|_\infty, \|h\|_1\right)$$

hence for each $h \in X$

$$\max\{\delta(h, Th), \delta(h, Sh)\} \preceq \varphi(Sh) - \varphi(Th)$$

Thus, all conditions of Theorem 3.5 are satisfied and T, S have a common fixed point $\overline{h}(x) = 0$.

4 Conclusion

In this article, motivated by Cho and Bae [47], we established some common fixed point theorems in the framework of cone-metric spaces with respect to strongly minihedral and continuous cone. The presented theorems can be considered as a new direction to prove common fixed point theorems using Caristi-Type mapping in cone metric spaces. We applied the above stated results to obtain a coupled fixed point theorem for two single valued mappings.

Acknowledgement

The authors would like to thank all the members of the Editorial Board who were responsible for dealing with this paper, and the anonymous referees for their valuable comments and suggestions, improving the content of this paper.

Competing Interests

Authors have declared that no competing interests exist.

References

[1] Ivar Ekeland. Sur les problemes variationnels. CR Acad. Sci. Paris Sér. AB. 1972;275:A1057-A1059.

[2] Ivar Ekeland. On the variational principle. Journal of Mathematical Analysis and Applications. 1974;47(2):324-353.

[3] Ivar Ekeland. Nonconvex minimization problems1. American Mathematical Society. 1979;1(3).

[4] Jean-Pierre Aubin, Hélène Frankowska. Set-valued analysis. Springer Science & Business Media; 2009.

[5] Bianchi M, Kassay G, Pini R. Existence of equilibria via Ekeland's principle. Journal of Mathematical Analysis and Applications. 2005;305(2):502-512.

[6] Wataru Takahashi. Nonconvex minimization theorems and fixed point theorems in complete metric spaces. Mathematica Japonica. 1996;44(2):381-391.

[7] Chr Tammer. A generalization of ekeland's variational principle. Optimization. 1992;25(2-3):129-141.

[8] A Göpfert, Chr Tammer, C Zălinescu. On the vectorial ekeland's variational principle and minimal points in product spaces. Nonlinear Analysis: Theory, Methods & Applications. 2000;39(7):909-922.

[9] Qiuxi Wang, Feng Chen, Fushan Huang. Maximum principle for optimal control problem of stochastic delay differential equations driven by fractional brownian motions. Optimal Control Applications and Methods; 2014.

[10] Lai-Jiu Lin, Mircea Balaj, Yu-Chia Ye. Quasi-variational relation problems and generalized ekeland's variational principle with applications. Optimization. 2014;63(9):1353-1365.

[11] Gong Xunhua. Ekeland's principle for set-valued vector equilibrium problems. Acta Mathematica Scientia. 2014;34(4):1179-1192.

[12] James Caristi. Fixed point theorems for mappings satisfying inwardness conditions. Transactions of the American Mathematical Society. 1976;215:241-251.

[13] James Caristi. Fixed point theory and inwardness conditions. Applied Nonlinear Analysis. 1979; 479-483.

[14] Stefan Banach. Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. Fund. Math. 1922;3(1):133-181.

[15] Lazaiz S, Chaira K, Aamri M, Marhrani El M. Some remarks on caristi type fixed point theorem. International Journal of Pure and Applied Mathematics. 2015;104(4):585-597.

[16] Abdul Latif, Nawab Hussain, Marwan A Kutbi. Applications of Caristi's fixed point results. Journal of Inequalities and Applications. 2012;1:1-12.

[17] Zhilong Li. Remarks on Caristi's fixed point theorem and Kirk's problem. Nonlinear Analysis: Theory, Methods & Applications. 2010;73(12):3751-3755.

[18] Kirk WA. Caristi's fixed point theorem and metric convexity. In Colloquium Mathematicae. 1976;1:81-86.

[19] Samih Lazaiz, Mohamed Aamri, Omar Zakary. Fixed point theorems for iterated mappings via caristi-type results. International Journal of Computer Applications. 2016;156(4):1-6.

[20] Farshid Khojasteh, Erdal Karapinar, Hassan Khandani. Some applications of caristi's fixed point theorem in metric spaces. Fixed Point Theory and Applications. 2016;1:1-10.

[21] Lakshmi Narayan Mishra, Shiv Kant Tiwari, Vishnu Narayan Mishra. Fixed point theorems for generalized weakly s-contractive mappings in partial metric spaces. J. Appl. Anal. Comput. 2015;5:600-612.

[22] Murthy PP, Mishra LN, Patel UD. n-tupled fixed point theorems for weak-contraction in partially ordered complete g-metric spaces. New Trends in Mathematical Sciences. 2015;3(4):50-75.

[23] Lakshmi Narayan Mishra, Shiv Kant Tiwari, Vishnu Narayan Mishra, Idrees A Khan. Unique fixed point theorems for generalized contractive mappings in partial metric spaces. Journal of Function Spaces. 2015;2015.

[24] Lakshmi Narayan Mishra, Mausumi Sen. On the concept of existence and local attractivity of solutions for some quadratic volterra integral equation of fractional order. Applied Mathematics and Computation. 2016;285:174-183.

[25] Subramanian Deepmala N, Mishra LN. The new generalized difference of χ_2 over p-metric spaces defined by musielak orlicz function. Journal of Progressive Research in Mathematics. 2016;9(1):1301-1311.

[26] Mishra LN, Agarwal RP, Sen M. Solvability and asymptotic behavior for some nonlinear quadratic integral equation involving erdélyi-kober fractional integrals on the unbounded interval, progr. Progr. Fract. Differ. Appl. 2016;2.

[27] Murthy PP, Lakshmi Narayan Mishra, Uma Devi Patel. Common fixed point theorems for generalized quadratic ($\psi 1$, $\psi 2$, φ)-weak contraction in complete metric spaces.

[28] Mishra LN. On existence and behavior of solutions to some nonlinear integral equations with applications. National Institute of Technology, Silchar 788 010, Assam, India.

[29] Zaid Mohammed Fadail, Abd Ghafur Bin Ahmad. Common fixed point theorem of n-order in cone metric spaces. Far East Journal of Mathematical Sciences. 2015;96(2):133-166.

[30] Zaid Mohammed Fadail, Abd Ghafur Bin Ahmad. Fixed point theorem of n-order in cone metric spaces. Far East Journal of Mathematical Sciences. 2013;79(1):49-76.

[31] Zaid Mohammed Fadail, Abd Ghafur Bin Ahmad. Some new results in cone metric spaces. In Proceedings of the 20th National Symposium on Mathematical Sciences: Research in Mathematical Sciences: A Catalyst for Creativity and Innovation, AIP Publishing. 2013;1522:896-902.

[32] Gerald Jungck. Commuting mappings and fixed points. The American Mathematical Monthly. 1976;83(4):261-263.

[33] Salvatore Sessa. On a weak commutativity condition of mappings in fixed point considerations. Publ. Inst. Math. 1982;32(46):149-153.

[34] Gerald Jungck. Compatible mappings and common fixed points. International Journal of Mathematics and Mathematical Sciences. 1986;9(4):771-779.

[35] Pant RP. Common fixed points of noncommuting mappings. Journal of Mathematical Analysis and Applications. 1994;188(2):436-440.

[36] Kannan R. Some results on fixed points–ii. The American Mathematical Monthly. 1969;76(4):405-408.

[37] Jungck G. Common fixed points for noncontinuous nonself maps on nonmetric Spaces; 1996.

[38] Jungck G, Rhoades BE. Fixed points for set valued functions without continuity; 1998.

[39] Long-Guang Huang, Xian Zhang. Cone metric spaces and fixed point theorems of contractive mappings. Journal of mathematical Analysis and Applications. 2007;332(2):1468-1476.

[40] Abbas M, Jungck G. Common fixed point results for noncommuting mappings without continuity in cone metric spaces. Journal of Mathematical Analysis and Applications. 2008; 341(1):416-420.

[41] Mujahid Abbas, Rhoades BE. Fixed and periodic point results in cone metric spaces. Applied Mathematics Letters. 2009;22(4):511-515.

[42] Dejan Ilić, Vladimir Rakočević. Common fixed points for maps on cone metric space. Journal of Mathematical Analysis and Applications. 2008;341(2):876-882.

[43] Dejan Ilić, Vladimir Rakočević. Quasi-contraction on a cone metric space. Applied mathematics letters. 2009;22(5):728-731.

[44] Klim D, Wardowski D. Dynamic processes and fixed points of set-valued nonlinear contractions in cone metric spaces. Nonlinear Analysis: Theory, Methods & Applications. 2009 71(11):5170-5175.

[45] Seung-Kab Yang, Jong-Sook Bae, Seong-Hoon Cho. Coincidence and common fixed and periodic point theorems in cone metric spaces. Computers & Mathematics with Applications. 2011;61(2):170-177.

[46] Seong-Hoon Cho, Jong-Sook Bae. Common fixed point theorems for mappings satisfying property (ea) on cone metric spaces. Mathematical and Computer Modelling. 2011;53(5):945-951.

[47] Seong-Hoon Cho, Jong-Sook Bae. Variational principles on cone metric spaces. Int. J. Pure Appl. Math. 2012;77(5):709-718.

[48] Seong-Hoon Cho, Jong-Sook Bae, Kwang-Soo Na. Fixed point theorems for multivalued contractive mappings and multivalued caristi type mappings in cone metric spaces. Fixed Point Theory and Applications. 2012;1:1-10.

Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)
http://sciencedomain.org/review-history/17663