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Necessary and Sufficient Conditions for Controllability of Double-delay Autonomous Linear Control Systems

Ukwu Chukwunenye[1*]

[1]*Department of Mathematics, University of Jos, P.M.B 2084, Jos, Nigeria.*

Author's contribution

The sole author designed, analyzed and interpreted and prepared the manuscript.

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ABSTRACT

This article formulated and proved necessary and sufficient conditions for the Euclidean controllability of double-delay autonomous linear control systems, in terms of rank conditions on the controllability matrices. The proof was achieved by the exploitation of the structure of the determining matrices, the relationship among the determining matrices, the indices of control systems and system's coefficients of the relevant system and an appeal to Taylor's theorem as applied to vector functions.

Keywords: Control systems; determining matrices; double-delay systems; Euclidean controllability; rank conditions; Taylor's theorem.

1. INTRODUCTION

Controllability results for multifarious and specific types of hereditary systems with diversity in treatment approaches are quite prevalent in control literature. [1] discussed Controllability of functional differential equations of retarded and neutral types to targets in function space; [2] obtained controllability conditions for systems with distributed delays in state and control;

**Corresponding author: E-mail: cukwu@hotmail.com;*

[3] formulated a necessary and sufficient condition for Euclidean controllability of system $\dot{x}(t) = Ax(t) + Bx(t-h) + Cu(t)$ with piecewise continuous controls using a sequence determining matrices for the free part of the above restricted system. Unfortunately, the investigation of the dependence of the controllability matrix for infinite horizon on that for finite horizon very crucial for his proof was not fully addressed. This problem has now been addressed by [4]; [5] obtained some criteria for function space controllability of linear neutral systems; [6] looked at controllability of nonlinear delay systems; [7] discussed controllability of nonlinear hereditary systems, using a fixed-point approach; [8] studied null controllability in function space of nonlinear neutral differential systems with limited controls; [9] discussed controllability of nonlinear systems with delays in both state and control using a constructive control approach and an appeal to Arzela-Ascoli, and Shauder fixed point theorems to guarantee the existence and admissibility of such controls; [10] investigated null controllability of nonlinear neutral differential equations; [11] developed computational criteria for the Euclidean controllability of the above delay system investigated by Gabasov and Kirillova, using the determining matrices with a very simple structure, effectively eliminating the afore-mentioned drawback. However, a major drawback of Ukwu's major result is that it relied on [12] for the necessary and sufficient conditions for the Euclidean controllability of the delay system , stated in terms of the control index matrices, which until [13] were a herculean or almost impossible task to obtain. Definitely, It would be a positive contribution to obtain computational criteria for Euclidean controllability of the more complex systems under consideration. Herein lies the justification for this investigation. It must be pointed out that the investigation is limited to autonomous systems. [14,15] studied controllability of Volterra Integro-differential systems.

In recent years, [16] formulated differential models and neutral systems for controlling the Wealth of Nations. His monograph derives from economic principles of the dynamics of national income, interest rate, employment, value of capital stock, prices and cumulative balances of payments. Chukwu used a Volterra neutral integro-differential game of pursuit where the quarry control is government intervention in the form of taxation, control of money and supply tariffs. Other relevant works by Chukwu in this area include [17] on Stability and time-optimal control of hereditary systems with application to the economic dynamics of the United States of America [18].

More research efforts on controllability include [19], where the author investigated the properties of cores for which the system with distributed delays in control is relatively controllable; [20]; [21]; where the authors established sufficient conditions for the controllability and null controllability of linear systems; Other notable results with focus on integro-differential equations and impulsive differential equations with finite and infinite delays include [22-28]. Some authors established sufficient conditions for the controllability and null controllability of linear systems using the variation of constant formula to deduce their controllability Grammian and exploiting the properties of the Grammian and the asymptotic stability of the free system, [29]. These works and others appropriate relevant Existence and Uniqueness of solutions theorems; the linear systems among the cited works use the qualitative properties of the indices of control systems or rank conditions to characterize controllability for the most part. The expressions for such indices were not determined.

2. MATERIALS AND METHODS

2.1 System of Interest

Consider the autonomous linear differential – difference control system of neutral type:

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-h) + A_2 x(t-2h) + B\,u(t);\ t \geq 0 \tag{1}$$

$$x(t) = \phi(t),\ t \in [-2h, 0],\ h > 0 \tag{2}$$

where h is a single scalar delay, A_0, A_1, A_2 are $n \times m$ constant matrices with real entries and B is an $n \times m$ constant matrix with the real entries. The initial function ϕ is in $C\left([-2h, 0], \mathbb{R}^n\right)$ equipped with sup norm. The control u is in $\Omega \subseteq L_\infty\left([0, t_1], \mathbb{R}^n\right)$. Such controls will be called admissible controls. $x(t), x(t-h), x(t-2h) \in \mathbb{R}^n$, for $t \in [0, t_1]$. If $x \in C([-2h, t_1], \mathbb{R}^n)$, then for $t \in [0, t_1]$ we define $x_t \in C([-2h, 0], \mathbb{R}^n)$ by $x_t(s) = x(t+s)$, $s \in [-2h, 0]$.

Let:

$$\hat{Q}_n(t_1) = [Q_0(s)B, Q_1(s)B, \cdots, Q_{n-1}(s)B : s \in [0, t_1), s = 0, h, \cdots, (n-1)h], \tag{3}$$

where $Q_k(s)$ is a determining matrix for the uncontrolled part of (1) and satisfies

$$Q_k(s) = A_0 Q_{k-1}(s) + A_1 Q_{k-1}(s-h) + A_2 Q_{k-1}(s-2h)$$

For $k = 0, 1, \ldots; s = 0, h, 2h, \ldots$ subject to $Q_0(0) = I_n$, the $n \times n$ identity matrix and $Q_k(s) = 0$ for $k < 0$ or $s < 0$.

Let r_0, r_1, r_2 be nonnegative integers and let $P_{0(r_0),1(r_1),2(r_2)}$ denote the set of all permutations of $\underbrace{0, 0, \cdots 0}_{r_0 \text{ times}}; \underbrace{1, 1, \cdots 1}_{r_1 \text{ times}}; \underbrace{2, 2, \cdots 2}_{r_2 \text{ times}}$: the permutations of the objects 0,1 and 2 in which i appears r_i times; $i \in \{0, 1, 2\}$. The following is proved in [30], among other alternative expressions for $Q_k(jh)$:

2.2 Theorem on $Q_k(jh)$

For $0 \le j \le k$, j, k integers, $k \ne 0$,

$$Q_k(jh) = \sum_{r=0}^{\left[\left[\frac{j}{2}\right]\right]} \sum_{(v_1, \cdots, v_k) \in P_{0(r+k-j),1(j-2r),2(r)}} A_{v_1} \cdots A_{v_k};$$

For $j \ge k \ge 1$, j, k integers,

$$Q_k(jh) = \begin{cases} \displaystyle\sum_{r=0}^{\left[\left[\frac{2k-j}{2}\right]\right]} \sum_{(v_1, \cdots, v_k) \in P_{0(r),1(2k-j-2r),2(r+j-k)}} A_{v_1} \cdots A_{v_k}, & 1 \le j \le 2k \\ 0, & j \ge 2k+1, \end{cases}$$

where $[[.]]$ denotes the greatest integer function.

2.3 Definition of Global Euclidean Controllability

The system (1) is said to be Euclidean controllable if for each $\phi \in C([-2h, 0], \mathbb{R}^n)$ defined by

$$\phi(s) = g(s), s \in [-2h, 0), \phi(0) = g(0) \in \mathbb{R}^n \tag{4}$$

and for each $x_1 \in \mathbb{R}^n$, there exists a t_1 and an admissible control $u \in \Omega$ such that the solution (response) $x(t, \phi, u)$ of (1) satisfies $x_0(\phi, u) = \phi$ and $x(t_1; \phi, u) = x_1$.

2.4 Definition of Euclidean Controllability on an Interval

Let $x(t,\phi,u)$ denote the solution of system (1) with initial function ϕ and admissible control u at time t. System (1) is said to be Euclidean controllable on the interval $[0,t_1]$, if for each ϕ in $C([-2h,0],\mathbb{R}^n)$ and $x_1\in\mathbb{R}^n$, there is an admissible control $u\in L_\infty([0,t_1],\mathbb{R}^n)$ such that $x_0(\phi,u)=\phi$ and $x(t_1,\phi,u)=x_1$. System (1) is Euclidean controllable if it is Euclidean controllable on every interval $[0,\ t_1],\ t_1>0$.

In the process of establishing necessary and sufficient conditions for the Euclidean controllability of system (1) on the interval $[0,t_1]$, the following lemma will be needed.

2.5 Lemma on Rank of Matrices

Let C be any n by nq matrix. Let η be an arbitrary n-dimensional column vector. Then C has full rank if and only if the equation $\eta^T C=0$ admits only the trivial (zero) solution.

Proof

Clearly $\operatorname{rank}[C]\le\min\{n,nq\}=n.$ If C has full rank, then C can be column- and row-reduced to a matrix of

the form $\begin{cases} I_n, & \text{if } q=1\\ \left[I_n, D_{n\times(n-1)q}\right], & \text{if } q>1\end{cases}$

Hence $\eta^T C=0\Leftrightarrow\eta=0$. Note that the equation $\eta^T D_{n\times(n-1)q}=0$ becomes redundant for $q>1$.
Suppose that C does not have full rank. Then $\operatorname{rank}[C]<n$. Let $\operatorname{rank}[C]=p$, for some integer $p<n$.
Two cases arise; <u>Case1:</u> $q=1$.

$q=1\Rightarrow C$ is reducible to a matrix of the form $\tilde{C}=\begin{bmatrix} I_p, & D_{p\times(n-p)}\\ 0_{(n-p)\times p}, & 0_{(n-p)\times(n-p)}\end{bmatrix}$, where $0_{(n-p)\times p}$ denotes a zero matrix with $n-p$ rows and p columns. Therefore solving the equation $\eta^T C=0$ is equivalent to solving $\eta^T\tilde{C}=0\Rightarrow\eta^T\begin{bmatrix} I_n\\ 0_{n-p,p}\end{bmatrix}\Rightarrow\eta_1=\eta_2=\cdots=\eta_p=0,$ and $\eta_{p+1},\cdots,\eta_n$ are arbitrary. Also letting $\tilde{\eta}=(\eta_1,\cdots,\eta_p)^T$, we see that $\eta^T\begin{bmatrix} D_{p\times(n-p)}\\ 0_{(n-p)\times(n-p)}\end{bmatrix}=0\Rightarrow\tilde{\eta}^T D_{p\times(n-p)}=0$ and $\eta_{p+1},\cdots,\eta_n$ are arbitrary.
Therefore the relation $\eta^T C=0$ does not imply that $\eta=0$.

<u>Case 2:</u> $q>1$.

$q>1\Rightarrow C$ is reducible to a matrix of the form $\tilde{\tilde{C}}=\begin{bmatrix} I_p, & D_{p\times(nq-p)}\\ 0_{(n-p)\times p}, & 0_{(n-p)\times(n-p)}\end{bmatrix}$; clearly $\eta^T\tilde{\tilde{C}}=0\Rightarrow\eta_1=\cdots\eta_p=0$ and $\eta_{p+1},\cdots,\eta_n$ are arbitrary. This completes the proof.

3. RESULTS AND DISCUSSION

3.1 Theorem on Rank Conditions for Euclidean Controllability of System (1)

Let $\hat{Q}_n(t_1)$ be defined as in (3). Then system (1) is Euclidean controllable $[0,t_1]$ if and only if $\operatorname{rank}\left[\hat{Q}_n(t_1)\right]=n.$ Moreover $\hat{Q}_n(t_1)$ and $\dim\hat{Q}_n(t_1)$ are expressible in the form

$$\hat{Q}(t_1)=\left[\; Q_k(s)B : k\in\{0,1,\cdots,n-1, s\in\{0,h,\cdots,\left(\min\left\{(n-1),\left[\left[\left[\frac{t_1-h}{h}\right]\right]\right]\right\}\right)h\;\right]$$

$$\mathrm{D}\operatorname{im}\left[\hat{Q}_n(t_1)\right]=n\times mn\left(\min\left\{\left[\left[\left[\frac{t_1}{h}\right]\right]\right],\ n\right\}\right)=n\times mn\left(1+\min\left\{\left[\left[\left[\frac{t_1-h}{h}\right]\right]\right],\ n-1\right\}\right).$$

Here $[[[.]]]$ denotes the least integer function, otherwise referred to as the ceiling function in Computer Science.

Proof

By theorem 2.3 of [31] and lemma 2.3 of [2], system (1) is Euclidean controllability on $[0,t_1]$ if and only if $c^{\mathrm{T}}X(\tau,t_1)B\neq 0$ for any $c\in\mathbb{R}^n$, $c\neq 0$, where $\tau\to X(\tau,t_1)$ denotes the control index matrix of (1) for fixed t_1.

Sufficiency**:** First we prove that if $\hat{Q}_n(t_1)=n,$ then (1) is Euclidean controllable on $[0,t_1]$. Equivalently we prove that if (1) is not Euclidean controllable on $[0,t_1]$, then $\operatorname{rank}\hat{Q}_n(t_1)<n$ because $\hat{Q}_n(t_1)$ has n rows and therefore has rank at most n. Suppose that system (1) is not Euclidean controllable on $[0,t_1]$. Then these exists a nonzero column vector $c\in\mathbb{R}^n$ such that:

$$c^{\mathrm{T}}X(\tau,t_1)B\equiv 0;\tau\in[0,t_1] \tag{5}$$

But:

$$X(\tau,t_1)\equiv 0\text{ , on }(t_1,\infty) \tag{6}$$

Therefore:

$$c^{\mathrm{T}}X(\tau,t_1)B\equiv 0,\text{ on }\ \tau\in[0,\infty] \tag{7}$$

yielding:

$$c^{\mathrm{T}}X^{(k)}\left((t_1-jh)^-,t_1\right)B=0,\ \ c^{\mathrm{T}}X^{(k)}\left((t_1-jh)^-,t_1\right)B=0 \tag{8}$$

for all integers $j:t_1-jh>0,\ k\in\{0,1,2,...\}.$

Now:

$$\Delta X^{(k)}(t_1-jh,t_1)=(-1)^k\ Q_k(jh), \tag{9}$$

for $j:t_1-jh>0,\ k\in\{0,1,2,...\}$, by theorem 3.1 of [4].

From (8) and (9) we deduce that:

$$(-1)^k c^{\mathrm{T}}\ Q_k(jh)=0 \tag{10}$$

for some $c\in\mathbb{R}^n, c\neq 0$, for all $j:t_1-jh>0,k\in\{0,1,2,...\}.$

By virtue of (3) and theorem 3.1 of [4], condition (10) implies that the nonzero vector c is orthogonal to all columns of $\hat{Q}_\infty(t_1)$ and hence orthogonal to all columns of $\hat{Q}_n(t_1)$. Thus $\hat{Q}_n(t_1)$ does not have full rank. Since $\hat{Q}_n(t_1)$ has n rows, we deduce that:

$$\operatorname{rank}\hat{Q}_n(t_1)<n \tag{11}$$

(11) proves the contra-positive statement: $\operatorname{rank}\hat{Q}_n(t_1)=n\Rightarrow$ (1) is Euclidean controllable on $[0,t_1].$

Necessity: Suppose that $\operatorname{rank}\hat{Q}_\infty(t_1) < n$. Then by lemma 2.5, $\exists c \in \mathbb{R}^n, c \neq 0$ such that:

$$c^{\mathrm{T}} Q_k(s) B = 0, \text{ for all}$$

$$s \in [0,\ t_1] \text{ and } k \in \{0,1,2,...\}. \tag{12}$$

From theorem 3.1 of [4],

$$\begin{aligned} 0 &= (-1)^k c^{\mathrm{T}} Q_k(jh) B = c^{\mathrm{T}} \Delta X^{(k)}(t_1 - jh, t_1) B \\ &= c^{\mathrm{T}} \left[X^{(k)}\left((t_1 - jh)^-, t_1\right) - X^{(k)}\left((t_1 - jh)^+, t_1\right) \right] B, \end{aligned} \tag{13}$$

for nonnegative integral $j: t_1 - jh > 0$. From (13), we deduce that:

$$c^{\mathrm{T}} X^{(k)}\left((t_1 - jh)^-, t_1\right) B = c^{\mathrm{T}} X^{(k)}\left((t_1 - jh)^+, t_1\right) B \tag{14}$$

for $k \in \{0,1,2,....\}$ and $j : t_1 - jh > 0$. (14) is equivalent to:

$$\psi^{(k)}\left(c, (t_1 - jh)^-\right) = \psi^{(k)}\left(c, (t_1 - jh)^+\right) \tag{15}$$

for $k \in \{0,1,2,....\}$ and $j : t_1 - jh > 0$. In particular, if $j = 0$, then (15) yields:

$$\psi^{(k)}\left(c, t_1^-\right) = \psi^{(k)}\left(c, t_1^+\right) \tag{16}$$

But:

$$\psi^{(k)}\left(c, t_1^+\right) = \lim_{\substack{\tau \to t_1 \\ \tau \in (t_1, t_1 + h)}} \psi^{(k)}(c, \tau) = \lim_{\substack{\tau \to t_1 \\ \tau \in (t_1, t_1 + h)}} c^{\mathrm{T}} X^{(k)}(\tau, t_1) B = 0, \tag{17}$$

Since $X^{(k)}(t_1, \tau) \equiv 0$ for all $\tau \in (t_1, \infty)$ and $k = 0,1,2,...$

Therefore:

$$\psi^{(k)}\left(c,\ t_1^-\right) = \psi^{(k)}\left(c,\ t_1^+\right) = 0 \tag{18}$$

for $k \in \{0,1,2,....\}$. In particular the left continuity of $X(t_1, \tau)$ at $\tau = t_1$ implies that of $\psi(c, \tau)$ at $\tau = t_1$.

Hence:

$$\psi(c, t_1) = \psi\left(c, t_1^-\right) \tag{19}$$

But:

$$\psi\left(c,\ t_1^-\right) = \psi\left(c,\ t_1^+\right) \tag{20}$$

$$\psi(c, t_1) = \psi\left(c, t_1^-\right) = \psi\left(c, t_1^+\right) = 0 \tag{21}$$

Since $\tau \to \psi(c, \tau)$ is piecewise analytic for $\tau \in (t_1 - (j+1)h, t_1 - jh)$, for all $j : t_1 - (j+1)h > 0$, we may apply Taylor's theorem to each component of $\psi(c, \tau)$ for the rest of the proof.

Set $a = t_1$. Now each component of the m-vector function $\psi(c, \tau)$ satisfies the hypothesis of Taylor's theorem, with $a = t_1$, because ψ (c, τ) is analytic on $(t_1 - (j+1)h,\ t_1 - jh)$, $j \in \{0,1,...\}$ such that $t_1 - (j+1)h > 0$. Denote the i^{th} component of $\psi^{(k)}(c, \tau)$ by $\psi_i^{(k)}(c, \tau)$; $i \in \{1,2,...,m\}$ Then by Taylor's theorem,

$$\psi_i(c, \tau) = \sum_{k=0}^{\infty} \frac{\psi_i^{(k)}\left(c, t_1^-\right)(\tau - t_1)^k}{k!} \tag{22}$$

for all $\tau \in (t_1 - h, t_1)$. From (21) we deduce that:

$$\psi_i(c, \tau) = 0 \tag{23}$$

for all $\tau \in (t_1 - h,\ t_1];\ \ i = 1,\ 2, ..., m$

Now set $a = t_1 - h,\ a - h = t_1 - 2h$. By (15) and (23) we deduce that:

$$\psi_i^{(k)}\left(c, (t_1 - h)^-\right) = \psi_i^{(k)}\left(c, (t_1 - h)^+\right) = 0 \tag{24}$$

By Taylor's theorem, applied on the τ-interval $(t_1 - 2h,\ t_1 - h)$:

$$\psi_i(c, \tau) = \sum_{k=0}^{\infty} \frac{\psi_i^{(k)}\left(c, (t_1 - h)^-\right)\left(\tau - (t_1 - h)\right)^k}{k!} \tag{25}$$

for $i \in \{1,2,\cdots,m\}$, for all $\tau \in (t_1 - 2h,\ t_1 - h)$. But $\psi_i\left(c, (t_1 - h)^-\right) = \psi_i(c,\ t_1 - h)$. Hence $\psi_i(c, \tau) \equiv 0$ on $(t_1 - 2h, t_1 - h)$. Continuing in the above fashion we get $\psi_i(c.\tau) = 0$, for all $\tau \in (0, h]$ for $i \in \{1,2,\cdots,m\}$. Finally we use the fact that $X(0,t_1) = X(0^+,t_1)$ to deduce that $\psi(c,0) = \psi(c,0^+) = 0.$

Hence $\psi(c,\tau) = 0,$ for all $t \in [0,t_1]$; that is, $\exists\ c \in \mathbb{R}^n, c \neq 0$ such that:

$$c^{\mathrm{T}} X(\tau,t_1) B \equiv 0 \quad \text{on} \quad [0,t_1] \tag{26}$$

We immediately invoke [11] to deduce that system (1) is not Euclidean controllable on $[0,t_1]$ for any $t_1 > 0$. This proves that if the system (1) is Euclidean controllable on $[0,t_1]$ then $\hat{Q}_\infty(t_1)$ attains its full rank, *n*. By theorem 3.1 of [4], $\operatorname{rank} \hat{Q}_\infty(t_1) = \operatorname{rank} \hat{Q}_n(t_1)$.

Hence:

$$\operatorname{rank} \hat{Q}_n(t_1) = n \tag{27}$$

Observe that for any given $t_1 > 0, \exists$ a non-negative integer $p : t_1 = ph + \sigma,$ for some $0 \le \sigma < h$;

thus $\left[\left[\left[\dfrac{t_1 - h}{h}\right]\right]\right] = \begin{cases} p, \ \sigma \neq 0 \\ p-1, \sigma = 0 \end{cases}$, proving the computable expression for $\left[\hat{Q}_n(t_1)\right]$.

The expression for the dimension follows from the fact that there are altogether $n\left(1+\min\left\{\left[\left[\left[\dfrac{t_1-h}{h}\right]\right]\right],\ n-1\right\}\right)$ column-wise concatenated matrices in $\hat{Q}_n(t_1)$, each of dimension $n \times m$.

This completes the proof of the theorem.

3.2 Illustrative Example of Theorem 3.1

$$\text{Let } A_0 = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 2 & 3 & 2 \end{pmatrix}, A_1 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & -1 & 1 \end{pmatrix}, A_2 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix}, B = \begin{pmatrix} 1 & -1 \\ 2 & 1 \\ 1 & 2 \end{pmatrix}, t_1 = 3,\ h = 0.5.$$

Then $m = 2$ and $n = 3.$ Recall that the controllability matrix is $\hat{Q}_n(t_1) = \left[Q_0(s)B, Q_1(s)B, \cdots, Q_{n-1}(s)B : s \in \{0, h, \cdots, \min\{t_1, (n-1)h\}\}\right],$ an n by $mn\left(1+\left[\left[\left[\min\left\{\dfrac{t_1}{h},\ n-1\right\}\right]\right]\right]\right)$ concatenated matrix of $n\left(1+\left[\left[\left[\min\left\{\dfrac{t_1}{h},\ n-1\right\}\right]\right]\right]\right)$ matrix product objects, each of dimension n by m. Clearly,

$$\begin{aligned} \hat{Q}_3(3) &= \left[Q_0(s)B, Q_1(s)B, \cdots, Q_2(s)B : s \in \{0, h, 2h\}\right] \\ &= \left[Q_0(0)B, Q_1(0)B, Q_2(0)B, Q_0(h)B, Q_1(h)B, Q_2(h)B, Q_0(2h)B, Q_1(2h)B, Q_2(2h)B\right] \end{aligned}$$

The rank is invariant if the controllability matrix is pruned, with the deletion of associated zero matrices. Consequently

$$\begin{aligned} \operatorname{rank} \hat{Q}_3(3) &= \operatorname{rank}\left[Q_0(0)B, Q_1(0)B, Q_2(0)B, Q_1(h)B, Q_2(h)B, Q_1(2h)B, Q_2(2h)B\right] \\ &= \operatorname{rank}\left[B, A_0B, A_0^2B, Q_1(h)B, Q_2(h)B, Q_1(2h)B, Q_2(2h)B\right]. \end{aligned}$$

The computational result of the controllability matrix $\hat{Q}_3(3)$ is as follows:

Columns 1 through 10

1	-1	6	3	20	10	4	2	40	26
2	1	2	1	2	1	5	4	41	25
1	2	10	5	38	19	4	5	53	37

Columns 11 through 18

253	152	13	11	207	256	1887	1269
179	94	17	16	230	166	1745	1093
401	247	13	14	253	200	2643	1836

By theorem 3.1, using the above parameters, the system (1) with initial function specification (2) is Euclidean controllable on the interval $[0,3]$.

4. CONCLUSION

This article pioneered the introduction of the least integer function in the statement and proof of the necessary and sufficient conditions for the Euclidean controllability of linear hereditary systems; this makes the controllability matrix in (3) quite computable and eliminates any ambiguity that could arise in its application. The proof relied on the results in [30,4], incorporated the characterization of Euclidean controllability in terms of the indices of control systems and appropriated Taylor's theorem as an indispensible tool. Finally, the article provided an illustrative example on the computation of the controllability matrices and stated the implication of the result for Euclidean controllability.

COMPETING INTERESTS

Author has declared that no competing interests exist.

REFERENCES

1. Banks HT, Kent GA. Control of functional differential equations of retarded and neutral type to targets in function space, SIAM J. Control. 1972;10:4.
2. Manitius A, Olbrot AW. Controllability conditions for systems with distributed delays in state and control. Archinum Autom. i. Telem. 1972;17:119-131.
3. Gabasov R, Kirillova F. The qualitative theory of optimal processes: Marcel Dekker Inc., New York; 1976.
4. Ukwu C. Rank Equality of controllability matrices of a class of double-delay control systems for finite and infinite horizons. Asian Journal of Mathematics and Computer Research. 2016;9(4):335-340.
5. Jacobs MQ, Langenhop. Criteria for function space controllability of linear neutral systems. SIAM J. Control Optim. 1976;14:1009-1048.
6. Dauer JP, Gahl RD. Controllability of nonlinear delay systems. J.O.T.A. 1977; 21:1.
7. Angell TS. On controllability for nonlinear hereditary systems; a fixed point approach. Nonlinear Analysis, Theory, Methods and Applications. 1980;4:529-548.
8. Onwuatu JU. On the null-controllability in function space of nonlinear systems of neutral functional differential equations with limited controls. J. Optimiz. Theory Appl. 1984;42:397–420.
9. Balachandran K. Controllability of Nonlinear Systems with Delays in both State and Control. Kybernetika. 1986; 22:4.
10. Underwood RG, Chukwu EN. Null Controllability of nonlinear neutral differential equations. J. Math. Analy. and Appl. 1988;129:474-483.
11. Ukwu C. Euclidean controllability and cores of euclidean targets for differential difference systems. Master of Science Thesis in Applied Math. with O.R. (Unpublished), North Carolina State University, Raleigh, N. C. U.S.A; 1992.
12. Manitius A. Control theory and topics in functional analysis. Int. Atomic Energy Agency, Vienna. 1978;3.
13. Ukwu C, Garba EJD. Derivation of an optimal expression for solution matrices for a class of double-delay differential systems. Journal of Mathematical Science. 2014a;25:1.
14. Balachandran K. Controllability of neutral volterra integro-differential systems. J. Austral. Math. Soc. Ser. B. 1992;34:18-25.

15. Balachandran K, Balasubramaniam P. A note on controllability of neutral volterra integro-differential systems. Journal of Applied Mathematics and Stochastic Analysis, Summer. 1993;6(2):153-160.
16. Chukwu EN. Differential models and neutral systems for controlling the wealth of nations. Series on Advances in Mathematics for Applied Sciences (Book 54), World Scientific Pub Co Inc; 1st Edition; 2001.
17. Chukwu EN. Stability and time-optimal control of hereditary systems with application to the economic dynamics of the US. Series: Stability and Control: Theory, Methods and Applications. CRC Press; 2002.
18. Chukwu EN. Optimal control of the growth of wealth of Nations series: Stability and Control: Theory, Methods and Applications. CRC Press; 2003.
19. Iheagwam VA. Characterization of Cores of targets for ordinary systems with distributed delays in control. JNMS. 2003;22:23-34.
20. Balachandran K, Anandhi ER. Controllability of neutral integro-differential infinite delay systems in Banach spaces. Taiwanese Journal of Mathematics. 2004;8(4):689-702.
21. Davis I, Jackreece P. Controllability and null controllability of linear systems. Journal of Applied Sciences and Environment Management. 2005;9(3): 31-36.
22. Chang YK, Chalishajar DN. Controllability of mixed Volterra-Fredholm type integro-differential inclusions in Banach spaces. J. Franklin Inst. 2008;345:499-507.
23. Chang YK, Anguraj A, Mallika AM. Controllability of impulsive neutral differential inclusions with infinite delay in Banach spaces. Chaos Solitons Fractals. 2009;39(4):1864-1876.
24. Balachandran K, Annapoorani N. Existence results for impulsive neutral evolution integro-differential equations with infinite delay. Nonlinear Anal. 2009;3:674-684.
25. Ye R. Existence of solutions for impulsive partial neutral functional differential equations with infinite delay. Nonlinear Anal. 2010;73:155-162.
26. Vijayakumar V, Sivasankaran S. Mallika AM. Existence of global solutions for second order impulsive abstract functional integro- differential equations. Dyn. Contin. Discrete Impuls. Syst. 2011;18:747–766.
27. Vijayakumar V, Sivasankaran S, Mallika AM. Existence of global solutions for second order impulsive neutral functional integro- differential equations with infinite delay. Nonlinear Stud. 2012;19(2):327–343.
28. Selvi S, Mallika AM. Controllability results for impulsive differential systems with finite delay. J. Nonlinear Sci. Appl. 2012;5: 2006–219.
29. Machado JA, Ravichandran C, Rivero M, Trujillo JJ. Controllability results for impulsive mixed – type functional integro-differential evolution equations with nonlocal conditions. Fixed Point Theory and Applications. 2013;2013:66.
30. Ukwu Chukwunenye. The structure of determining matrices for a class of double-delay control systems. International Journalof Mathematics and Statistics Invention (IJMSI). 2014;2(3):14-30.
31. Chukwu EN. Stability and time-optimal control of hereditary systems. Academic Press, New York; 1992.

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