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The Cauchy Problem for the Camassa-Holm Equation with Quartic Nonlinearity in Besov Spaces

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Author's contribution

The sole author designed, analyzed and interpreted and prepared the manuscript.

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Abstract

In this paper, we study the Camassa-Holm equation with quartic nonlinearity. We prove that the Cauchy problem for this equation is locally well-posed in the critical Besov space $B^{3/2}_{2,1}$ or in $B^s_{p,r}$ with $1 \leq p, r \leq +\infty$, $s > \max\{1+1/p, 3/2\}$. We also prove that if a weaker $B^q_{p,r}$-topology is used, then the solution map becomes Hölder continuous. Furthermore, if the space variable x is taken to be periodic, we show that the solution map defined by the associated periodic boundary problem is not uniformly continuous in $B^s_{2,r}$ with $1 \leq r \leq +\infty, s > 3/2$ or $r = 1, s = \frac{3}{2}$.

Keywords: Camassa-Holm equation; qurtic nonlinearity; well-posedness; Besov spaces; non-uniform dependence.

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1 Introduction

The well-known Camassa-Holm equation (CH for short)

$$u_t - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \tag{1.1}$$

was first derived by Fokas and Fuchssteiner in [1] for studying completely integrable generalization of the KdV equation with bi-Hamiltonian structure, and later proposed by Camassa and Holm in [2] as a model for unidirectional propagation of shallow water waves over a flat bottom, and it is proved that the CH equation is completely integrable and possesses an infinite number of conservation laws. It is quite different from the KdV equation that the CH equation has peakon solution and breaking waves, see [2]-[4]. As noted in [5], it is intriguing to find both phenomena of soliton interaction and wave breaking can exhibit in one mathematical model of shallow water waves. The CH equation is well investigated in view of mathematical point and a lot of achievements had been made. For instance, the Cauchy problem for CH and periodic CH equation were studied in [6]-[8], the global weak solutions and global conservative and dissipative solutions were obtained in [9]-[13], the peakon and smooth solitary wave solutions were proved to be orbital stable and interacts like solitons [14]-[17], on the wave-breaking we refer to [18]-[22].

There are considerable researches having studied the following generalized CH equation [23]−[28],

$$u_t - u_{xxt} + au^n u_x = 2u_x u_{xx} + uu_{xxx}, \tag{1.2}$$

they have focused on the stronger nonlinear convection, that is, the nonlinear convection term uu_x in (1.1) has been changed to $u^n u_x$ in (1.2), which makes the structure of their solutions change a lot. There are many new nonlinear phenomena arise for equation (1.2), such as compacton solitons with compact support, solitons with cusps, or peakons, cf. [29]−[37]. Four simple ansäzs were proposed to obtain abundant solutions: compactons, solitary patterns solutions having infinite slopes or cusps, and solitary waves in [29]. By using bifurcation method, peakons and periodic cusp waves were studied in [33]-[35], the explicit expressions of peakons for (1.2) are given in some special cases. In [36] some new exact peaked solitary waves were derived. By employing polynomial ansäz the periodic wave and peaked solitary waves of equation (1.2) were investigated in [37]. The Cauchy problem of equation (1.2) was studied in [38, 39] and the solitary waves for the equation (1.2) was proved to be orbital stable for any speed in [23, 40] as $n = 2, 3$.

Motivated by the previous work on the generalized CH equation, in this paper we aim to study the Cauchy problem of equation (1.2) with the initial value date in Besov space and the continuity properties of the solution map in the case $n = 3$,

$$u_t - u_{xxt} + au^3 u_x = 2u_x u_{xx} + uu_{xxx}, \tag{1.3}$$

$$u(0, x) = u_0(x) \in B^s_{p,r}, \quad x \in \mathbb{R},\ t \in \mathbb{R}^+. \tag{1.4}$$

The main difficulties we are encountered with are to prove that the approximate solutions to the Cauchy problem (1.3) (1.4) is uniformly bounded in $B^{1/2}_{2,1}$ and that the approximate solutions is a Cauchy sequence in $B^{1/2}_{2,1}$ due to the fact that (1.3) involving quartic nonlinearity is much more complicated than cubic or square nonlinearities. For the method, it is more convenient to reformulate (1.3) (1.4) as the nonlocal form:

$$\begin{cases} \partial_t u + u\partial_x u + \partial_x (1 - \partial_x^2)^{-1} \left(-\frac{a}{4}u^4 + \frac{1}{2}u^2 - \frac{1}{2}(u_x)^2 \right) = 0, \\ u(0, x) = u_0(x) \in B^s_{p,r}, \quad x \in \mathbb{R},\ t \in \mathbb{R}^+. \end{cases} \tag{1.5}$$

Denoting $A = (1 - \partial_x^2)^{-1}$ and

$$F(u) = \partial_x A(-\frac{a}{4}u^4 + \frac{1}{2}u^2 - \frac{1}{2}(u_x)^2), \tag{1.6}$$

then (1.5) can be rewrite as follows:

$$\begin{cases} \partial_t u + u\partial_x u + F(u) = 0, & x \in \mathbb{R},\ t \in \mathbb{R}^+, \\ u(0,x) = u_0(x) \in B^s_{p,r}, \end{cases} \tag{1.7}$$

It is easy to verify that $F : B^s_{p,r} \to B^s_{p,r}$ is continuous when $s > 1+\frac{1}{p}$ with $1 \le p, r \le \infty$. Therefore, it is reasonable to expect the existence of local solutions in $B^s_{p,r}$ with $s > 1 + \frac{1}{p}$, $1 \le p, r \le \infty$. However, we will need the products estimate of the type $\|fg\|_{B^{s-1}_{p,r}} \le \|f\|_{B^{s-1}_{p,r}}\|g\|_{B^{s-2}_{p,r}}$, hence the condition $s > 3/2$ is required.

Before starting the main results, we explain the notations and conventions used throughout this paper.

Notations. In this paper we adopt the following notations:
The notation $\lesssim$ denotes the estimates that hold up to some universal constant which may change from line to line but whose meaning is clear from the context. $\mathscr{S}(\mathbb{R})$ is the space of rapidly decreasing functions on $\mathbb{R}$ and $\mathscr{S}'(\mathbb{R})$ is its dual space. If the function spaces are over $\mathbb{R}$, we will drop $\mathbb{R}$ in our notations of function spaces if there is no ambiguity. When the function spaces are over $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, we will **not** omit $\mathbb{T}$ in our notations.

For $T > 0$, $s \in \mathbb{R}$ and $1 \le p \le \infty$, we define

$$E^s_{p,r}(T) = \begin{cases} C([0,T]; B^s_{p,r}) \cap C^1([0,T]; B^{s-1}_{p,r}), & \text{if } r < \infty, \\ L^\infty(0,T; B^s_{p,\infty}) \cap Lip([0,T]; B^{s-1}_{p,\infty}), & \text{if } r = \infty. \end{cases}$$

Our main results are as follows:

Theorem 1.1. *If $u_0 \in B^{3/2}_{2,1}$, then we have*

(1) *There is a $T_{u_0} > 0$ such that the problem (1.5) has a unique solution u in $E^{3/2}_{2,1}(T_{u_0})$ which depends continuously on the initial data u_0. Moreover, the solution u satisfies*

$$\|u(t)\|_{B^{3/2}_{2,1}} \le 2\|u_0\|_{B^{3/2}_{2,1}} + 1, \quad \text{for} \quad 0 \le t \le T_{u_0} \le \frac{7}{48C\left(\|u_0\|_{B^{3/2}_{2,1}} + 1\right)^3}, \tag{1.8}$$

where $C > 0$ is a constant.

(2) *The solution map $u_0 \mapsto u(t,\cdot)$ defined by (1.5) is Hölder continuous as a map from $B(0,R) \subset B^{3/2}_{2,1}$, with the $B^{1/2}_{2,\infty}$ topology, to $C([0,\widetilde{T}], B^q_{2,1})$, $\frac{1}{2} < q < \frac{3}{2}$. More precisely, if $u_0, v_0 \in B(0,R) \subset B^{3/2}_{2,1}$ and u, v are the solutions corresponding to the initial data u_0, v_0, respectively, then for $\frac{1}{2} < q < \frac{3}{2}$, we have*

$$\|u(t) - v(t)\|_{B^q_{2,1}} \lesssim \|u_0 - v_0\|^\alpha_{B^{1/2}_{2,\infty}}, \quad 0 \le t \le \widetilde{T}, \tag{1.9}$$

where $\widetilde{T} = \frac{7}{48C(R+1)^3}$, and $\alpha = \left(\frac{3}{2} - q\right)\exp\{-C_R\}$ with the constant $C_R > 0$.

Theorem 1.2. *Let $1 \le p, r \le \infty$, $s \ge \max\{1 + \frac{1}{p}, \frac{3}{2}\}$ and $u_0 \in B^s_{p,r}$, then we have*

(1) *There is a $T_{u_0} > 0$ such that the problem (1.5) has a unique solution u in $E^s_{p,r}(T_{u_0})$. The solution map $u_0 \mapsto u$ defined by the problem (1.5) is continuous from $B^s_{p,r}$ into $E^{s'}_{p,r}(T_{u_0})$ for every $s' < s(s' \le s \quad if \quad r < \infty)$. Moreover, the solution u satisfies*

$$\|u(t)\|_{B^s_{p,r}} \le 2\|u_0\|_{B^s_{p,r}} + 1, \quad \text{for} \quad 0 \le t \le T_{u_0} \le \frac{7}{48C(\|u_0\|_{B^s_{p,r}} + 1)^3}, \tag{1.10}$$

where $C>0$ is a constant depending on p,r,s.

(2) *If $u_0, v_0 \in B(0,R) \subset B^s_{p,r}$. Let u,v be two solutions corresponding to the initial data u_0, v_0, respectively. Then we have*

$$\|u(t)-v(t)\|_{B^{q_2}_{p,r}} \lesssim \|u_0-v_0\|^{\alpha}_{B^{q_1}_{p,r}}, \quad 0\le t\le \widetilde{T}, \tag{1.11}$$

where $\widetilde{T}=\frac{1}{48C(R+1)^3}$, q_1, q_2 and the exponent α satisfy

$$\alpha=\begin{cases} 1, & \text{if } s\neq 2+\frac{1}{p},\ \max\{\frac{1}{2},\ \frac{1}{p}\}<q_1=q_2\le s-1,\ q_1=q_2\neq 1+\frac{1}{p}, \\ s-q_2, & \text{if } s\neq 2+\frac{1}{p},\ s-1=q_1<q_2<s, \\ q_1-\frac{1}{p}, & \text{if } s=2+\frac{1}{p},\ \max\{\frac{1}{2},\ \frac{1}{p}\}<q_1<q_2=1+\frac{1}{p}, \end{cases} \tag{1.12}$$

In other words, the solution map $u_0\mapsto u(t,\cdot)$ defined by (1.5) *is Hölder continuous as a map from $B(0,R)\subset B^s_{p,r}$, with the $B^{q_1}_{p,r}$ topology, to $C([0,\widetilde{T}], B^{q_2}_{p,r})$.*

Remark 1.1. When the space variable x is taken to be periodic with period 2π, we consider the periodic boundary value problem instead of the Cauchy problem (1.5), that is

$$\begin{cases} \partial_t u+u\partial_x u+F(u)=0,\ t>0, \quad \gamma\neq 0, \\ u(0,x)=u_0(x)\in B^s_{p,r}, \quad\ x\in\mathbb{T}=\mathbb{R}/2\pi\mathbb{Z}. \end{cases} \tag{1.13}$$

When $x\in\mathbb{T}=\mathbb{R}/2\pi\mathbb{Z}$, since Theorems 1.1 and 1.2 also hold for $x\in\mathbb{T}$, we know that the solution map is continuous in $B^s_{p,r}$ and in $B^{3/2}_{2,1}$ (even Hölder continuous if some suitable weak topology is chosen). However, when $x\in\mathbb{T}$, from the following theorem, we see that it is not uniformly continuous in $B^s_{2,r}$ with $1\le r\le\infty$ and $s>3/2$ or $r=1$ and $s=3/2$.

Theorem 1.3. *When $x\in\mathbb{T}=\mathbb{R}/2\pi\mathbb{Z}$, for the solution map $u_0\mapsto u(t,\cdot)$ defined by* (1.13)*, we have the following results:*

(1) *When $1\le r\le\infty$ and $s>3/2$, the map $u_0\mapsto u(t,\cdot)$ is not uniformly continuous from any bounded subset in $B^s_{2,r}$ into $C([0,T];B^s_{2,r}(\mathbb{T}))$ if $r<\infty$ or into $L^\infty(0,T;B^s_{2,\infty}(\mathbb{T}))$ if $r=\infty$ for some $T>0$. More precisely, there exist two sequences of solutions $u_i(t)$ and $v_i(t)$ such that*

$$\|u_i(t)\|_{B^s_{2,r}(\mathbb{T})}+\|v_i(t)\|_{B^s_{2,r}(\mathbb{T})}\lesssim 1, \tag{1.14}$$

$$\lim_{i\to\infty}\|u_i(0)-v_i(0)\|_{B^s_{2,r}(\mathbb{T})}=0, \tag{1.15}$$

$$\liminf_{i\to\infty}\|u_i(t)-v_i(t)\|_{B^s_{2,r}(\mathbb{T})}\gtrsim|\sin t|, \quad 0\le t\le T. \tag{1.16}$$

(2) *When $s=3/2$, the map $u_0\mapsto u(t,\cdot)$ is not uniformly continuous from any bounded subset in $B^{3/2}_{2,1}$ into $C([0,\widehat{T}];B^{3/2}_{2,1}(\mathbb{T}))$ for some $\widehat{T}>0$. More precisely, there exist two sequences of solutions $u_i(t)$ and $v_i(t)$ such that*

$$\|u_i(t)\|_{B^{3/2}_{2,1}(\mathbb{T})}+\|v_i(t)\|_{B^{3/2}_{2,1}(\mathbb{T})}\lesssim 1, \tag{1.17}$$

$$\lim_{i\to\infty}\|u_i(0)-v_i(0)\|_{B^{3/2}_{2,1}(\mathbb{T})}=0, \tag{1.18}$$

$$\liminf_{i\to\infty}\|u_i(t)-v_i(t)\|_{B^{3/2}_{2,1}(\mathbb{T})}\gtrsim|\sin t|, \quad 0\le t\le \widehat{T}. \tag{1.19}$$

2 Preliminaries

In this section, we shall recall some basic facts on the Littlewood–Paley theory, Besov spaces and the transport equations theory that will be used in this paper. We refer to [41]-[45] for the details of them.

Let χ, ϕ be two functions satisfying $\chi, \phi \in C_c^\infty(\mathbb{R}), 0 \le \chi, \phi \le 1$, $\chi(\xi) = 1$ for $|\xi| \le \frac{3}{4}$, supp$\chi = \{\xi \in \mathbb{R}, |\xi| \le \frac{4}{3}\}$ and $\phi(\xi) = \chi(2^{-1}\xi) - \chi(\xi)$. Then

$$\chi(\xi) + \sum_{j\in\mathbb{N}} \phi(2^{-j}\xi) = 1, \quad \forall \xi \in \mathbb{R},$$
$$\text{supp } \phi(2^{-j}\cdot) \bigcap \text{supp } \phi(2^{-j'}\cdot) = \varnothing \quad \text{if} \quad |j - j'| \ge 2,$$
$$\text{supp } \chi(\cdot) \bigcap \text{supp } \phi(2^{-j}\cdot) = \varnothing \quad \text{if} \quad j \ge 1.$$

For $u \in \mathscr{S}'(\mathbb{R})$, we define the nonhomogeneous dyadic block operators as

$$\Delta_{-1}u = \chi(D)u = \mathcal{F}_x^{-1}\chi\mathcal{F}_x u,$$
$$\Delta_j u = \phi(2^{-j}D)u = \mathcal{F}_x^{-1}\phi(2^{-j}\xi)\mathcal{F}_x u, \quad \text{if} \quad j \ge 0,$$

where $\mathcal{F}_x u$ is the Fourier transform in x. Then we have

$$u = \sum_{j=-1}^{\infty} \Delta_j u \quad \text{converges in } \mathscr{S}'(\mathbb{R}) \text{ or in } H^s(\mathbb{R}).$$

We define the low frequency cut-off S_j as $S_j u = \sum_{i=-1}^{j-1} \Delta_i u$. Direct computation implies that for any $1 \le p \le \infty$,

$$\Delta_i\Delta_j u \equiv 0, \quad \text{if} \quad |i - j| \ge 2, \quad \forall\, u, v \in \mathscr{S}'(\mathbb{R}),$$
$$\Delta_j(S_{i-1}u\Delta_i v) \equiv 0, \quad \text{if} \quad |i - j| \ge 5, \quad \forall\, u, v \in \mathscr{S}'(\mathbb{R}),$$
$$\|\Delta_i u\|_{L^p} \le C\|u\|_{L^p}, \quad \forall\, u \in L^p(\mathbb{R}),$$
$$\|S_j u\|_{L^p} \le C\|u\|_{L^p}, \quad \forall\, u \in L^p(\mathbb{R}).$$

Definition 2.1 (Besov spaces)**.** Let $s \in \mathbb{R}$, $1 \le p, r \le +\infty$. The nonhomogeneous Besov space $B^s_{p,r}(\mathbb{R})$ is defined by

$$B^s_{p,r}(\mathbb{R}) = \left\{ f \in \mathscr{S}'(\mathbb{R}) : \|f\|_{B^s_{p,r}(\mathbb{R})} < \infty \right\},$$

where $\|f\|_{B^s_{p,r}(\mathbb{R})} = \|2^{js}\Delta_j f\|_{l^r(L^p)} = \left\| \left(2^{js}\|\Delta_j f\|_{L^p}\right)_{j\ge -1} \right\|_{l^r}$.

Remark 2.1. When $p = 2$ and $1 \le r \le +\infty$, the above definition is equivalent to the following:

$$\|u\|_{B^s_{2,r}(\mathbb{R})} = \begin{cases} \left(\left\| (1+\xi^2)^{\frac{s}{2}}\mathcal{F}_x u \right\|^r_{L^2(-1,1)} + \sum\limits_{j\in\mathbb{N}} \left\| (1+\xi^2)^{\frac{s}{2}}\mathcal{F}_x u \right\|^r_{L^2(C_j)} \right)^{\frac{1}{r}}, & r < \infty, \\ \sup\limits_{j\in\mathbb{N}} \left\{ \left\| (1+\xi^2)^{\frac{s}{2}}\mathcal{F}_x u \right\|_{L^2(-1,1)}, \left\| (1+\xi^2)^{\frac{s}{2}}\mathcal{F}_x u \right\|_{L^2(C_j)} \right\}, & r = \infty, \end{cases}$$

where $C_j = \{\xi \in \mathbb{R} : 2^j \le |\xi| \le 2^{j+1}\}$. In particular, if $p = r = 2$, then $B^s_{2,2} = H^s$.

The following Lemma summarizes some useful properties of $B^s_{p,r}$.

Lemma 2.1 ([41]-[45]). *Let* $s \in \mathbb{R}$, $1 \le p,\, r,\, p_j,\, r_j \le \infty$, $j = 1, 2$, *then*

(1) $B^s_{p,r}(\mathbb{R})$ is a Banach space and is continuously embedded in $\mathscr{S}'(\mathbb{R})$.

(2) $B^{s_1}_{p_1,r_1} \hookrightarrow B^{s_2}_{p_2,r_2}$ if $p_1 \le p_2$; $r_1 \le r_2$ and $s_2 = s_1 - (\frac{1}{p_1} - \frac{1}{p_2})$.
$B^{s_1}_{p,r_1} \hookrightarrow B^{s_2}_{p,r_2}$ is locally compact if $s_2 < s_1$; $r_1 \le r_2$.

(3) $\forall s > 0$, $B^s_{p,r} \cap L^\infty$ is a Banach algebra. $B^s_{p,r}$ is a Banach algebra $\Leftrightarrow B^s_{p,r} \hookrightarrow L^\infty \Leftrightarrow s > \frac{1}{p}$ (or $s \ge \frac{1}{p}$ and $r = 1$).

(4) $\forall \theta \in [0,1]$, $s = \theta s_1 + (1-\theta)s_2$,

$$\|f\|_{B^s_{p,r}} \le C\|f\|^{\theta}_{B^{s_1}_{p,r}} \|f\|^{1-\theta}_{B^{s_2}_{p,r}}, \quad \forall\, f \in B^{s_1}_{p,r} \cap B^{s_2}_{p,r}.$$

(5) $\forall \theta \in (0,1)$, $s_1 > s_2$, $s = \theta s_1 + (1-\theta)s_2$,

$$\|u\|_{B^s_{p,1}} \le \frac{C(\theta)}{s_1 - s_2} \|u\|^{\theta}_{B^{s_1}_{p,\infty}} \|u\|^{1-\theta}_{B^{s_2}_{p,\infty}}, \quad \forall\, u \in B^{s_1}_{p,\infty}.$$

(6) If $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $B^s_{p,r}$ and u_n converges to u in $\mathscr{S}'(\mathbb{R})$, then $u \in B^s_{p,r}$ and

$$\|u\|_{B^s_{p,r}} \le \liminf_{n\to\infty} \|u_n\|_{B^s_{p,r}}.$$

Remark 2.2. Recall that we denote $P(D) = (1 - \partial_x^2)^{-1}$ and we shall use the estimates

$$\|P(D)f\|_{B^s_{p,r}} \lesssim \|f\|_{B^{s-2}_{p,r}}, \quad \|\partial_x P(D)f\|_{B^s_{p,r}} \lesssim \|f\|_{B^{s-1}_{p,r}}.$$

In terms of the operator, we say that $P(D) = (1 - \partial_x^2)^{-1}$ is an isometry from $B^s_{p,r}$ into $B^{s+2}_{p,r}$ and $\partial_x P(D)$ is an isometry from $B^s_{p,r}$ into $B^{s+1}_{p,r}$.

Now we recall some results in Besov spaces $B^s_{p,r}$ of the transport equation. We refer to [42, 43, 46, 47] for the details.

Lemma 2.2 (*a priori* estimates). *Let* $1 \le p, r \le \infty$ *and* $s > -\min\left\{\frac{1}{p}, 1 - \frac{1}{p}\right\}$. *Assume that* $f_0 \in B^s_{p,r}$, $F \in L^1(0,T; B^s_{p,r})$ *and* $\partial_x v \in L^1(0,T; B^{s-1}_{p,r})$ *if* $s > 1 + \frac{1}{p}$ *or* $\partial_x v \in L^1(0,T; B^{1/p}_{p,r} \cap L^\infty)$ *otherwise. If* $f \in L^\infty(0,T; B^s_{p,r}) \cap C([0,T]; \mathscr{S}'(\mathbb{R}))$ *solves the following 1–D linear transport equation:*

$$f_t + v f_x = F, \quad t > 0, \quad x \in \mathbb{R}, \tag{2.1}$$

$$f(0,x) = f_0, \quad x \in \mathbb{R}, \tag{2.2}$$

then there exists a constant C *depending only on* s, p, r *such that the following statements hold:*

(1) *For all* $t \in [0,T]$,

$$\|f\|_{B^s_{p,r}} \le \|f_0\|_{B^s_{p,r}} + \int_0^t \|F(\tau)\|_{B^s_{p,r}} \mathrm{d}\tau + C\int_0^t V'(\tau)\|f(\tau)\|_{B^s_{p,r}} \mathrm{d}\tau, \tag{2.3}$$

and hence,

$$\|f\|_{B^s_{p,r}} \le e^{CV(t)} \left(\|f_0\|_{B^s_{p,r}} + \int_0^t e^{-CV(\tau)} \|F(\tau)\|_{B^s_{p,r}} \mathrm{d}\tau \right), \tag{2.4}$$

where

$$V(t) = \begin{cases} \int_0^t \|v_x(\tau)\|_{B^{1/p}_{p,r} \cap L^\infty} \mathrm{d}\tau, & \text{if } s < 1 + \frac{1}{p}, \\ \int_0^t \|v_x(\tau)\|_{B^{s-1}_{p,r}} \mathrm{d}\tau, & \text{if } s > 1 + \frac{1}{p} \text{ or } \left(s = 1 + \frac{1}{p} \text{ and } r = 1\right). \end{cases}$$

(2) *If* $f = v$, *then* (1) *holds true for all* $s > 0$ *with* $V(t) = \int_0^t \|v_x(\tau)\|_{L^\infty} \mathrm{d}\tau$.

Lemma 2.3 (Existence and uniqueness). *Let p, r, s, f_0 and F be as in the statement of Lemma 2.2. Assume that $v \in L^\rho(0,T;B^{-M}_{\infty,\infty})$ for some $\rho > 1$, $M > 0$ and $v_x \in L^1(0,T;B^{s-1}_{p,r})$ if $s > 1+\frac{1}{p}$ or $s = 1+\frac{1}{p}$, $r = 1$ and $v_x \in L^1(0,T;B^{1/p}_{p,\infty} \cap L^\infty)$ if $s < 1+\frac{1}{p}$. Then (2.1) (2.2) have a unique solution $f \in L^\infty(0,T;B^s_{p,r}) \cap C([0,T];B^{s'}_{p,1})$ for any $s' < s$ and the inequalities of Lemma 2.2 hold true. Moreover, if $r < \infty$, then $f \in C([0,T];B^s_{p,r})$.*

We will use the following estimates frequently.

Lemma 2.4 (Moser-type estimates, see [41, 42, 43, 48]). *Letting $1 \le p, r \le +\infty$, then we have the following estimates:*

(i) *For $s > 0$, $\|fg\|_{B^s_{p,r}} \le C\left(\|f\|_{B^s_{p,r}}\|g\|_{L^\infty} + \|f\|_{L^\infty}\|g\|_{B^s_{p,r}}\right) \quad \forall\, f, g \in B^s_{p,r} \cap L^\infty$.*
(ii) *For all $s_1 \le \frac{1}{p} < s_2$ ($s_2 \ge \frac{1}{p}$ if $r = 1$) and $s_1 + s_2 > 0$,*

$$\|fg\|_{B^{s_1}_{p,r}} \le C\|f\|_{B^{s_1}_{p,r}}\|g\|_{B^{s_2}_{p,r}} \quad \forall\, f \in B^{s_1}_{p,r}, \quad g \in B^{s_2}_{p,r}.$$

Lemma 2.5 ([46]). *For any $f \in B^{-1/2}_{2,1}$, $g \in B^{1/2}_{2,1}$, there holds the product estimate*

$$\|fg\|_{B^{-1/2}_{2,\infty}} \lesssim \|f\|_{B^{-1/2}_{2,1}}\|g\|_{B^{1/2}_{2,1}}.$$

Lemma 2.6 ([43]). *There is a constant $C > 0$ such that for $s \in \mathbb{R}$, $\varepsilon > 0$ and $1 \le p \le \infty$,*

$$\|f\|_{B^s_{p,1}} \le C\frac{1+\varepsilon}{\varepsilon}\|f\|_{B^s_{p,\infty}} \ln\left(\mathrm{e} + \frac{\|f\|_{B^{s+\varepsilon}_{p,\infty}}}{\|f\|_{B^s_{p,\infty}}}\right), \qquad \forall\, f \in B^{s+\varepsilon}_{p,\infty}.$$

Now we recall the Osgood lemma cf. [41, 49].

Lemma 2.7 (Osgood Lemma,[41, 49]). *Let $\rho \ge 0$ be a measurable function, $\gamma > 0$ be a locally integrable function and μ be a continuous and increasing function. Assume that, for some nonnegative real number c, the function ρ satisfies*

$$\rho(t) \le c + \int_{t_0}^t \gamma(t')\mu(\rho(t'))\, \mathrm{d}t'.$$

If $c > 0$, then $-\mathcal{M}(\rho(t)) + \mathcal{M}(c) \le \int_{t_0}^t \gamma(t')\, \mathrm{d}t'$ with $\mathcal{M} = \int_x^1 \frac{\mathrm{d}r}{\mu(r)}$.
If $c = 0$ and μ verifies the condition $\int_0^1 \frac{\mathrm{d}r}{\mu(r)} = +\infty$, then the function $\rho = 0$.

For given solutions u and v, we will need to estimate the difference $F(u) - F(v)$ in terms of $u - v$. We now summarize the estimate as follows:

Lemma 2.8. *If $u, v \in B^{3/2}_{2,1}$, we have*

$$\|F(u) - F(v)\|_{B^{1/2}_{2,\infty}} \lesssim \left(\|u\|^3_{B^{3/2}_{2,1}} + \|v\|^3_{B^{3/2}_{2,1}} + \|u\|_{B^{3/2}_{2,1}} + \|v\|_{B^{3/2}_{2,1}}\right)\|u - v\|_{B^{1/2}_{2,1}}$$

Proof. By the expression of $F(u)$ and Lemma 2.5, we have

$$\begin{aligned}\|F(u)-F(v)\|_{B^{1/2}_{2,\infty}} &\lesssim \|u^4-v^4\|_{B^{-1/2}_{2,\infty}}+\|u^2-v^2\|_{B^{-1/2}_{2,\infty}}+\|u_x^2-v_x^2\|_{B^{-1/2}_{2,\infty}}\\ &\lesssim \left(\|u\|^3_{B^{3/2}_{2,1}}+\|u\|^2_{B^{3/2}_{2,1}}\|v\|_{B^{3/2}_{2,1}}+\|u\|_{B^{3/2}_{2,1}}\|v\|^2_{B^{3/2}_{2,1}}+\|v\|^3_{B^{3/2}_{2,1}}\right)\|u-v\|_{B^{1/2}_{2,1}}\\ &\quad+2\left(\|u\|_{B^{3/2}_{2,1}}+\|v\|_{B^{3/2}_{2,1}}\right)\|u-v\|_{B^{1/2}_{2,1}}\\ &\lesssim \left(\|u\|^3_{B^{3/2}_{2,1}}+\|v\|^3_{B^{3/2}_{2,1}}+\|u\|_{B^{3/2}_{2,1}}+\|v\|_{B^{3/2}_{2,1}}\right)\|u-v\|_{B^{1/2}_{2,1}}\end{aligned}$$

□

Lemma 2.9 ([50, 51]). *Let $\sigma,\alpha\in\mathbb{R}$. If $\lambda\in\mathbb{Z}^+$ and $\lambda\gg1$, then*

$$\|\sin(\lambda x-\alpha)\|_{B^\sigma_{2,r}(\mathbb{T})}=\|\cos(\lambda x-\alpha)\|_{B^\sigma_{2,r}(\mathbb{T})}=\begin{cases}2^{\frac{1}{r}}\pi(1+\lambda^2)^{\sigma/2}\approx\lambda^\sigma, & r<\infty,\\ \pi(1+\lambda^2)^{\sigma/2}\approx\lambda^\sigma, & r=\infty.\end{cases}$$

3 Local Well-posedness

In this section, we prove Theorems 1.1 and 1.2. We will give a detailed proof for The Theorem 1.1 and in the final remark, we give the modifications which is needed to prove Theorem 1.2. To begin with, we construct the approximate solutions.

3.1 Approximate solution

Starting from $u_1=0$ and by induction, we define a sequence of smooth functions $\{u_n\}, n\in\mathbb{N}$ by solving the following transport equation iteratively:

$$\partial_t u_{n+1}+u_n\partial_x u_{n+1}=-F(u_n), \tag{3.1}$$

$$u_{n+1}(0,x)=S_{n+1}u_0, \tag{3.2}$$

where $F(u)$ is given in (1.6). Since all the data belong to $B^\infty_{2,1}$, from Lemma 2.3 and by induction, we obtain that for all $n\geq1$, the above equation has a global solution u_{n+1} belonging to $C(\mathbb{R}^+;B^\infty_{2,1})$.

3.2 Uniform bounds of the approximate solutions

For $n\in\mathbb{N}$, let $U_n(t)=\int_0^t\|u_n\|_{B^{3/2}_{2,1}}\mathrm{d}\tau$. Since $\|u_{n+1}(0,x)\|_{B^{3/2}_{2,1}}\lesssim\|u_0\|_{B^{3/2}_{2,1}}$, from the estimate (2.4) of Lemma 2.2, we have

$$\begin{aligned}\|u_{n+1}(t)\|_{B^{3/2}_{2,1}} &\leq e^{CU_n(t)}\left(\|u_0\|_{B^{3/2}_{2,1}}+C\int_0^t e^{-CU_n(\tau)}\|F(u_n(\tau))\|_{B^{3/2}_{2,1}}\,\mathrm{d}\tau\right)\\ &\leq e^{CU_n(t)}\left(\|u_0\|_{B^{3/2}_{2,1}}+C\int_0^t e^{-CU_n(\tau)}\|\partial_x A(-au^4/4+u^2/2-u_x^2/2)(\tau)\|_{B^{3/2}_{2,1}}\,\mathrm{d}\tau\right)\\ &\leq e^{CU_n(t)}\left(\|u_0\|_{B^{3/2}_{2,1}}+C\int_0^t e^{-CU_n(\tau)}\left(\|u_n(\tau)\|^4_{B^{3/2}_{2,1}}+\|u_n(\tau)\|^2_{B^{3/2}_{2,1}}\right)\mathrm{d}\tau\right)\\ &\leq e^{CU_n(t)}\left(\|u_0\|_{B^{3/2}_{2,1}}+C\int_0^t e^{-CU_n(\tau)}\left(\|u_n(\tau)\|_{B^{3/2}_{2,1}}+1\right)^4\mathrm{d}\tau\right).\end{aligned} \tag{3.3}$$

Let $T > 0$ such that $6C\left(\|u_0\|_{B^{3/2}_{2,1}}+1\right)^3 T < 1$, we claim that for any $n \in \mathbb{N}$,

$$\|u_n\|_{B^{3/2}_{2,1}} \leq \frac{\|u_0\|_{B^{3/2}_{2,1}}+1}{\left(1-6C\left(\|u_0\|_{B^{3/2}_{2,1}}+1\right)^3 t\right)^{1/3}} - 1, \quad \forall\, t \in [0,T]. \tag{3.4}$$

Assume (3.4) is true for n. We now prove that it also holds true for $n+1$. Since $U_n(t) = \int_0^t \|u_n\|_{B^{3/2}_{2,1}} \mathrm{d}\tau$, by (3.4), we have

$$\begin{aligned}
e^{CU_n(t)-CU_n(\tau)} &\leq \exp\left\{C\int_\tau^t \|u_n\|_{B^{3/2}_{2,1}}\,\mathrm{d}t'\right\} \leq \exp\left\{C\int_\tau^t \left(\|u_n\|_{B^{3/2}_{2,1}}+1\right)^3\,\mathrm{d}t'\right\} \\
&\leq \exp\left\{C\int_\tau^t \frac{\left(\|u_0\|_{B^{3/2}_{2,1}}+1\right)^3}{\left(1-6C\left(\|u_0\|_{B^{3/2}_{2,1}}+1\right)^3 t'\right)}\,\mathrm{d}t'\right\} \\
&= \left(\frac{1-6C\left(\|u_0\|_{B^{3/2}_{2,1}}+1\right)^3 \tau}{1-6C\left(\|u_0\|_{B^{3/2}_{2,1}}+1\right)^3 t}\right)^{1/6}.
\end{aligned} \tag{3.5}$$

From the above equation, we see clearly that when $\tau = 0$, $U_n(0) = 0$, we have

$$\mathrm{e}^{CU_n(t)} \leq \frac{1}{\left(1-6C\left(\|u_0\|^3_{B^{3/2}_{2,1}}+1\right)^3 t\right)^{1/6}}. \tag{3.6}$$

Using (3.3), (3.4), (3.5) and (3.6) gives rise to

$$\begin{aligned}
\|u_{n+1}(t)\|_{B^{3/2}_{2,1}}+1 &\leq e^{CU_n(t)}\left(\|u_0\|_{B^{3/2}_{2,1}}+1+C\int_0^t e^{-CU_n(\tau)}\left(\|u_n(\tau)\|_{B^{3/2}_{2,1}}+1\right)^4\,\mathrm{d}\tau\right) \\
&\leq \frac{\|u_0\|_{B^{3/2}_{2,1}}+1}{\left(1-6C\left(\|u_0\|_{B^{3/2}_{2,1}}+1\right)^3 t\right)^{1/6}}\left[1-\frac{1}{6}\int_0^t \frac{\mathrm{d}\left(1-6C\left(\|u_0\|_{B^{3/2}_{2,1}}+1\right)^3 \tau\right)}{\left(1-6C\left(\|u_0\|_{B^{3/2}_{2,1}}+1\right)^3 \tau\right)^{7/6}}\right] \\
&\leq \frac{\|u_0\|_{B^{3/2}_{2,1}}+1}{\left(1-6C\left(\|u_0\|_{B^{3/2}_{2,1}}+1\right)^3 t\right)^{1/3}}.
\end{aligned}$$

Hence (3.4) is true and $\{u_n\}$ is uniformly bounded in $C([0,T]; B^{3/2}_{2,1})$ for all $t \in [0,T]$. Setting $T_{u_0} = \frac{7}{48C(\|u_0\|_{B^{3/2}_{2,1}}+1)^3}$, for $n \in \mathbb{N}$, we can conclude that the solution u_n exists for $0 \leq t \leq T_{u_0}$ and satisfies the bound

$$\|u_n(t)\|_{B^{3/2}_{2,1}} \leq 2\|u_0\|_{B^{3/2}_{2,1}}+1 \lesssim 1, \qquad 0 \leq t \leq T_{u_0}. \tag{3.7}$$

Furthermore, using the equation (3.1) yields that for $0 \le t \le T_{u_0}$,

$$\|\partial_t u_{n+1}\|_{B^{1/2}_{2,1}} \le \|u_n\|_{B^{3/2}_{2,1}} \|\partial_x u_{n+1}\|_{B^{1/2}_{2,1}} + \|F(u_n)\|_{B^{3/2}_{2,1}} \lesssim 1. \tag{3.8}$$

Hence we conclude that $\{u_n\} \subset E^{3/2}_{2,1}(T_{u_0})$ is uniformly bounded.

3.3 Convergence of the approximate solutions

We now show that $\{u_n\}$ convergences in $C([0, T_{u_0}]; B^{1/2}_{2,\infty})$. For $m, n \in \mathbb{N}$, from (3.1) we have

$$\begin{aligned}\partial_t \left(u_{n+m+1} - u_{n+1}\right) &+ u_{n+m}\partial_x \left(u_{n+m+1} - u_{n+1}\right) \\ &= \left(u_n - u_{n+m}\right) \partial_x u_{n+1} + \left[F(u_n) - F(u_{n+m})\right].\end{aligned} \tag{3.9}$$

Applying Lemma 2.2 to the above equation, we have that for $w_{n+1,m} = u_{n+m+1} - u_{n+1}$ and $0 < t < T_{u_0}$, there holds

$$\|w_{n+1,m}(t)\|_{B^{1/2}_{2,\infty}} \le e^{CU_{n+m}(t)} \left(\|w_{n+1,m}(0)\|_{B^{1/2}_{2,\infty}} + C\int_0^t e^{-CU_{n+m}(\tau)} \|H^{n,m}(\tau)\|_{B^{1/2}_{2,\infty}} d\tau \right).$$

Here $U_{n+m}(t) = \int_0^t \|u_{n+m}\|_{B^{3/2}_{2,\infty}} \mathrm{d}\tau \le \int_0^t \|u_{n+m}\|_{B^{3/2}_{2,1}} \mathrm{d}\tau \lesssim 1$ due to (3.7),

$$\|H^{n,m}(\tau)\|_{B^{1/2}_{2,\infty}} = \|\left(u_n - u_{n+m}\right) \partial_x u_{n+1} + \left[F(u_n) - F(u_{n+m})\right]\|_{B^{1/2}_{2,\infty}}.$$

Using Lemma 2.5, Lemma 2.6, Lemma 2.8 and (3.7), we find that for some $M > \|u_0\|_{B^{3/2}_{2,1}}$, there holds

$$\|H^{n,m}(\tau)\|_{B^{1/2}_{2,\infty}} \lesssim \|u_n - u_{n+m}\|_{B^{1/2}_{2,\infty}} \lesssim \|w_{n,m}\|_{B^{1/2}_{2,\infty}} \ln\left(\mathrm{e} + \frac{M}{\|w_{n,m}\|_{B^{1/2}_{2,\infty}}}\right).$$

Since

$$\|w_{n+1,m}(0)\|_{B^{1/2}_{2,\infty}} \lesssim \|(u_{n+m+1} - u_{n+1})(0)\|_{B^{1/2}_{2,1}} \le C2^{-n}\|u_0\|_{B^{3/2}_{2,1}}.$$

we have

$$\|w_{n+1,m}(t)\|_{B^{1/2}_{2,\infty}} \lesssim 2^{-n} + C\int_0^t \|w_{n,m}\|_{B^{1/2}_{2,\infty}} \ln\left(\mathrm{e} + \frac{M}{\|w_{n,m}\|_{B^{1/2}_{2,\infty}}}\right) d\tau.$$

Let $H_{n,m}(t) = \|w_{n+1,m}(t)\|_{B^{1/2}_{2,\infty}}$. From the above estimate, we obtain that

$$H_{n+1,m}(t) \lesssim 2^{-n} + \int_0^t H_{n,m}(\tau) \ln\left(\mathrm{e} + \frac{1}{H_{n,m}(\tau)}\right) \,\mathrm{d}\tau, \quad t \in [0, T_{u_0}].$$

As the function $x\ln(\mathrm{e} + \frac{1}{x})$ is nondecreasing, we see that $H_n(t) \triangleq \sup_{m\in\mathbb{N}} H_{n,m}(t)$ satisfies

$$H_{n+1}(t) \lesssim 2^{-n} + \int_0^t H_n(\tau) \ln\left(\mathrm{e} + \frac{1}{H_n(\tau)}\right) \,\mathrm{d}\tau, \quad t \in [0, T_{u_0}].$$

Let $\widetilde{H}(t) = \limsup_{n\to\infty} H_n(t)$, taking upper limits both side to the above equation with respect to n we have

$$\widetilde{H}(t) \lesssim \int_0^t \widetilde{H}(\tau) \ln\left(\mathrm{e} + \frac{1}{\widetilde{H}(\tau)}\right) \,\mathrm{d}\tau.$$

Since $x\ln\left(\mathrm{e}+\frac{1}{x}\right)$ is nondecreasing and $\displaystyle\int_0^1 \frac{1}{x\ln\left(\mathrm{e}+\frac{1}{x}\right)}\,\mathrm{d}x = +\infty$, we can infer from Lemma 2.7 that $\widetilde{H}(t) = 0$, in other words, $\{u_n\}$ is a Cauchy sequence in $C([0,T_{u_0}];B^{1/2}_{2,\infty})$. By (5) in Lemma 2.1, we see

$$\|u_{n+m}-u_n\|_{B^1_{2,1}} \lesssim \|u_{n+m}-u_n\|^{\frac{1}{2}}_{B^{1/2}_{2,\infty}}\|u_{n+m}-u_n\|^{\frac{1}{2}}_{B^{3/2}_{2,\infty}} \lesssim \|u_{n+m}-u_n\|^{\frac{1}{2}}_{B^{1/2}_{2,\infty}}.$$

Hence $\{u_n\}_{n\in\mathbb{N}}$ is actually a Cauchy sequence in $C([0,T];B^1_{2,1})$ and therefore $\{u_n\}$ converges to some function $u \in C([0,T];B^1_{2,1})$.

3.4 Existence and regularity of the solution

Since $\|u_n\|_{B^{3/2}_{2,1}}$ is uniformly bounded by $2\|u_0\|_{B^{3/2}_{2,1}}+1$, the property (6) in Lemma 2.1 guarantees that $\|u\|_{B^{3/2}_{2,1}} \le 2\|u_0\|_{B^{3/2}_{2,1}}+1$, which means that $u \in L^\infty(0,T_{u_0};B^{3/2}_{2,1})$. If $s'<1$, we have

$$\|u_n-u\|_{B^{s'}_{2,1}} \le \|u_n-u\|_{B^1_{2,1}}. \tag{3.10}$$

If $1<s'<3/2$, by interpolation again, we have

$$\|u_n-u\|_{B^{s'}_{2,1}} \le \|u_n-u\|^\theta_{B^1_{2,1}}\|u_n-u\|^{1-\theta}_{B^{3/2}_{2,1}} \lesssim \|u_n-u\|^\theta_{B^1_{2,1}}\|u_0\|^{1-\theta}_{B^{3/2}_{2,1}}, \tag{3.11}$$

where $\theta = 3-2s'$. From (3.10) and (3.11), we see that $u_n \to u$ in $C([0,T];B^{s'}_{2,1})$ for all $s'<3/2$. Taking limits to (3.1) (3.2), we can deduce that u indeed solves (1.5). Since $r=1<\infty$, from Lemma 2.3 and (3.3), we know that $u\in C([0,T_{u_0}];B^{3/2}_{2,1})$ and $u_t \in C([0,T_{u_0}];B^{1/2}_{2,1})$.

3.5 Uniqueness of the solution and Hölder continuity of the solution map

The continuity of the solution map can be obtained by following the same steps in [46] and we omit the details in this paper. Now we focus on the uniqueness of the solution and the Hölder continuity of the solution map. To show this, we need the following Lemma.

Lemma 3.1. *If $u,v \in C([0,T];\mathscr{D}'(\mathbb{T}))\bigcap L^\infty(0,T;B^{3/2}_{2,1})$ are two solutions to (1.5) for some $T>0$ with initial data $u_0,v_0 \in B^{3/2}_{2,1}$ respectively, then for any $s' \in (\frac{1}{2},\frac{3}{2})$ and $t\in[0,T]$, we have*

$$\|u-v\|_{B^{s'}_{2,1}} \lesssim \|u_0-v_0\|^{\theta\exp\{-CT\}}_{B^{1/2}_{2,\infty}}, \quad \theta = \frac{3}{2}-s' \in (0,1], \tag{3.12}$$

where $C>0$ is a constant depending on $\|u\|_{L^\infty(0,T;B^{3/2}_{2,1})}$ and $\|v\|_{L^\infty(0,T;B^{3/2}_{2,1})}$.

Proof. Since u,v are two solutions to (1.5) with initial data $u_0,v_0\in B^{3/2}_{2,1}$, respectively, we find that $w=u-v$ satisfies

$$w_t + v\partial_x w = -w\partial_x u - (F(u)-F(v)), \quad w(0,x) = u_0-v_0. \tag{3.13}$$

Let $U(t) = \int_0^t \|v_x(\tau)\|_{B^{1/2}_{2,\infty}\cap L^\infty}\,\mathrm{d}\tau$, by Lemma 2.2, we have

$$\|w(t)\|_{B^{1/2}_{2,\infty}} \le \|w_0\|_{B^{1/2}_{2,\infty}}e^{CU(t)} + \int_0^t e^{CU(t)-CU(\tau)}\|\widehat{F}\|_{B^{1/2}_{2,\infty}}\,\mathrm{d}\tau,$$

where $\widehat{F} = -(u-v)\partial_x u - [F(u)-F(v)]$. Since $u, v \in L^\infty(0,T;B^{3/2}_{2,1})$, the estimate of $u-v$ is essential in the derivation of (3.9), we obtain that for all $t \in [0,T]$,

$$\|w(t)\|_{B^{1/2}_{2,\infty}} \lesssim \|w(0)\|_{B^{1/2}_{2,\infty}} + C\int_0^t \|w(\tau)\|_{B^{1/2}_{2,\infty}} \ln\left(\mathrm{e} + \frac{M}{\|w(\tau)\|_{B^{1/2}_{2,\infty}}}\right) \mathrm{d}\tau. \tag{3.14}$$

Since $\ln\left(\mathrm{e}+\frac{M}{x}\right) \le \ln(\mathrm{e}+1)(1-\ln\frac{x}{M})$ for $x \in (0,M]$, from (3.14), we have

$$\frac{\|w(t)\|_{B^{1/2}_{2,\infty}}}{M} \lesssim \frac{\|w(0)\|_{B^{1/2}_{2,\infty}}}{M} + C\int_0^t \frac{\|w\|_{B^{1/2}_{2,\infty}}}{M}\left(1-\ln\frac{\|w\|_{B^{1/2}_{2,\infty}}}{M}\right)\mathrm{d}\tau, \quad \forall\, t\in[0,T].$$

Thanks to Lemma 2.7, we obtain

$$\frac{\|w(t)\|_{B^{1/2}_{2,\infty}}}{\mathrm{e}M} \lesssim \left(\frac{\|w(0)\|_{B^{1/2}_{2,\infty}}}{\mathrm{e}M}\right)^{\exp\{-CT\}}.$$

Therefore, we have

$$\|w(t)\|_{B^{1/2}_{2,\infty}} \lesssim \|w(0)\|^{\exp\{-CT\}}_{B^{1/2}_{2,\infty}}. \tag{3.15}$$

If $\frac{1}{2} < s' < \frac{3}{2}$, by interpolation, the embedding $B^{3/2}_{2,1} \hookrightarrow B^{3/2}_{2,\infty}$, we arrive at

$$\|w\|_{B^{s'}_{2,1}} \le \|w\|^{\theta}_{B^{1/2}_{2,\infty}} \|w\|^{1-\theta}_{B^{3/2}_{2,\infty}} \le \|w\|^{\theta}_{B^{1/2}_{2,\infty}}\left(\|u\|_{B^{3/2}_{2,1}} + \|v\|_{B^{3/2}_{2,1}}\right)^{1-\theta} \lesssim \|w\|^{\theta}_{B^{1/2}_{2,\infty}},$$

where $\theta = \frac{3}{2} - s'$. Using (3.15), we obtain that

$$\|w\|_{B^{s'}_{2,1}} \lesssim \|w\|^{\theta}_{B^{1/2}_{2,\infty}} \lesssim \|w(0)\|^{\theta\exp\{-CT\}}_{B^{1/2}_{2,\infty}}. \tag{3.16}$$

We finish the proof. □

Obviously, the uniqueness of solution is a corollary of Lemma 3.1. Furthermore, for any initial data $u_0 \in B(0,R) \subset B^{3/2}_{2,1}$, by (1.8), we see that the lifespan T_u of the corresponding solution u to (1.5) satisfies

$$T_u > \frac{7}{48C\left(\|u_0\|_{B^{3/2}_{2,1}} + 1\right)^3} > \frac{7}{48C\,(R+1)^3} \triangleq \widetilde{T},$$

where $\widetilde{T}$ does not depend on u. Therefore, we can find a $\widetilde{T} > 0$ such that for all $u_0 \in B(0,R) \subset B^{3/2}_{2,1}$, the corresponding solution $u \in C([0,\widetilde{T}]; B^{3/2}_{2,1})$. Directly from (1.8) and Lemma 3.1, (1.9) is proved and hence we complete the proof of Theorem 1.1.

Remark 3.1. The proof for Theorem 1.2 is much easier than the one for Theorem 1.1. We can also use the standard iterative process to obtain a solution. One can obtain the uniform bounds for the approximate solutions $\{u_n\}$, and then prove that $\{u_n\}$ is a Cauchy sequence in $B^{s-1}_{p.r}$ by using the same method as in [43]. Similarly, we can verify that the limit function u is a solution and $u \in E^s_{p.r}$. Furthermore, continuity and Hölder continuity of the solution map can be obtained by following the steps in [50]. The details for Theorem 1.2 are omitted.

4 Non-uniform Dependence

In this section, we will show that the solution map defined by (1.13) is not uniformly continuous. We argue in two cases:

$$1 \le r \le \infty,\ s > 3/2 \ \text{ and } \ r = 1, s = 3/2.$$

For the first case, we can follow the same steps as in [50] to prove the (1) in Theorem 1.3 and hence we only give the proof for the critical case.

4.1 Approximate solutions and actual solutions

Following the approach in [52], we choose approximate solutions as

$$u^{k,n}(t,x) = kn^{-1} + n^{-3/2}\cos\theta, \quad \text{where } \ \theta = nx - kt,\ n \in \mathbb{Z}^+,\ k = -1, 1. \tag{4.1}$$

Substituting $u^{k,n}$ into (1.13) gives rise to the error which is defined to be

$$E = \partial_t u^{k,n} + u^{k,n}\partial_x u^{k,n} + F(u^{k,n}), \tag{4.2}$$

Lemma 4.1. *Let E be as in (4.2), $\sigma = 1, \infty$. Then we have $\|E\|_{B^{1/2}_{2,\sigma}} \lesssim n^{-3/2}$.*

Proof. $\|E\|_{B^{1/2}_{2,\sigma}} \le \left\|\partial_t u^{k,n} + u^{k,n}\partial_x u^{k,n}\right\|_{B^{1/2}_{2,\sigma}} + \left\|F(u^{k,n})\right\|_{B^{1/2}_{2,\sigma}}$. Via direct computation and Lemma 2.9, we have

$$\left\|\partial_t u^{k,n} + u^{k,n}\partial_x u^{k,n}\right\|_{B^{1/2}_{2,\sigma}} \lesssim \left\|-\frac{1}{2}n^{-2}\sin(2nx - 2kt)\right\|_{B^{1/2}_{2,\sigma}} \lesssim n^{-3/2}, \tag{4.3}$$

$$\begin{aligned}
\|F(u^{k,n})\|_{B^{1/2}_{2,\sigma}} &\lesssim \left\|(u^{k,n})^3(\partial_x u^{k,n})\right\|_{B^{-3/2}_{2,\sigma}} + \left\|u^{k,n}(\partial_x u^{k,n})\right\|_{B^{-3/2}_{2,\sigma}} \\
&\quad + \left\|(\partial_x u^{k,n})(\partial_x^2 u^{k,n})\right\|_{B^{-3/2}_{2,\sigma}} \\
&\lesssim \left\|(k^2n^{-2} + n^{-3}/2)(-n^{-3/2}k\sin\theta + \frac{1}{2}n^{-2}\sin 2\theta)\right\|_{B^{-3/2}_{2,\sigma}} \\
&\quad + \left\|-k^2n^{-4}\sin 2\theta + \frac{1}{2}kn^{-7/2}(\sin 3\theta + \sin\theta)\right\|_{B^{-3/2}_{2,\sigma}} \\
&\quad + \left\|\frac{1}{4}kn^{-9/2}(\sin 3\theta - \sin\theta) + \frac{1}{8}n^{-5}\sin 4\theta\right\|_{B^{-3/2}_{2,\sigma}} \\
&\quad + \left\|-kn^{-3/2}\sin\theta - \frac{1}{2}n^{-2}\sin 2\theta\right\|_{B^{-3/2}_{2,\sigma}} \\
&\quad + \left\|\frac{1}{2}\sin 2\theta\right\|_{B^{-3/2}_{2,\sigma}} \\
&\lesssim n^{-3/2}.
\end{aligned} \tag{4.4}$$

We will complete the proof by putting the estimates (4.3) and (4.4) together. ☐

Via Lemma 2.9, for $n \gg 1$ and any $t \geqslant 0$, we have

$$\|u^{k,n}(t,x)\|_{B^{3/2}_{2,\sigma}} = \|kn^{-1} + n^{-3/2}\cos(nx - kt)\|_{B^{3/2}_{2,\sigma}} \lesssim 1, \quad \sigma = 1, \infty. \tag{4.5}$$

4.2 Estimating the differences between approximate solutions and actual solutions

We consider the following periodic boundary value problem with initial data $u^{k,n}(0,x)$, i.e.

$$\begin{cases} u_t + u\partial_x u = -F(u), \; t \in \mathbb{R}^+, \quad x \in \mathbb{T}, \\ u(0,x) = u^{k,n}(0,x) = kn^{-1} + n^{-3/2}\cos(nx) \in B^{3/2}_{2,1}. \end{cases} \tag{4.6}$$

From Theorem 1.1, let $u_{k,n} \in C([0,T]; B^{3/2}_{2,1})$ be the unique solution to (4.6), where

$$T = \frac{3}{48C\left(\|u^{k,n}(0)\|_{B^{3/2}_{2,1}} + 1\right)^3} > 0$$

can be chosen independent of n by (4.5). Then (1.8) turns out to be

$$\|u_{k,n}(t,x)\|_{B^{3/2}_{2,1}} \lesssim \|u^{k,n}(0,x)\|_{B^{3/2}_{2,1}} \lesssim 1, \quad 0 \le t \le T, \; n \in \mathbb{N}. \tag{4.7}$$

Letting $v = u^{k,n} - u_{k,n}$ and $F' = E - \partial_x u_{k,n} v - \left[F\left(u^{k,n}\right) - F\left(u_{k,n}\right)\right]$, then v satisfies

$$\begin{cases} \partial_t v + u^{k,n}\partial_x v = F', \quad t \in \mathbb{R}^+, \; x \in \mathbb{T}, \\ v(0,x) = 0, \qquad x \in \mathbb{T}. \end{cases} \tag{4.8}$$

We now estimate the $B^{1/2}_{2,\infty}$ norm and $B^{5/2}_{2,\infty}$ norm of v.

Lemma 4.2. *When $n \gg 1$, for some constant $C > 0$, we have*

$$\|v(t)\|_{B^{1/2}_{2,\infty}} \lesssim n^{-\frac{3}{2}\exp\{-CT\}}, \quad \|v(t)\|_{B^{5/2}_{2,\infty}} \lesssim n, \quad \text{for} \quad 0 \le t \le T, \tag{4.9}$$

Proof. By Lemma 2.2 and $v(0) = 0$, we have

$$\|v(t)\|_{B^{1/2}_{2,\infty}} \le C\int_0^t \exp\left\{C\int_\tau^t \left\|\partial_x u^{k,n}(\tau')\right\|_{B^{1/2}_{2,\infty}\cap L^\infty} d\tau'\right\} \cdot \left\|F'\right\|_{B^{1/2}_{2,\infty}} d\tau,$$

Note that $\|\partial_x u_{k,n} v\|_{B^{1/2}_{2,\infty}} \lesssim \|u_{k,n}\|_{B^{3/2}_{2,1}} \|v\|_{B^{1/2}_{2,1}} \lesssim \|v\|_{B^{1/2}_{2,1}}$. From (4.5), (4.7), Lemma 2.8, the above estimate and Lemma 4.1, we see that

$$\|v(t)\|_{B^{1/2}_{2,\infty}} \lesssim \int_0^t \|v\|_{B^{1/2}_{2,1}} \,\mathrm{d}\tau + \int_0^t n^{-3/2} \,\mathrm{d}\tau.$$

Applying Lemma 2.6, (4.5) and (4.7) yields

$$\|v(t)\|_{B^{1/2}_{2,\infty}} \lesssim n^{-3/2}T + C\int_0^t \|v(\tau)\|_{B^{1/2}_{2,\infty}} \ln\left(\mathrm{e} + \frac{1}{\|v(\tau)\|_{B^{1/2}_{2,\infty}}}\right) \mathrm{d}\tau.$$

Repeat the process as in Lemma 3.1, we obtain that for some constant $C > 0$,

$$\|v(t)\|_{B^{1/2}_{2,\infty}} \lesssim n^{-\frac{3\exp\{-CT\}}{2}}. \tag{4.10}$$

To estimate the $B^{5/2}_{2,\infty}$ norm of v, we first estimate the $B^{5/2}_{2,\infty}$ norm of $u_{k,n}(t)$. For each fixed $n \in \mathbb{N}$, applying the property (2) in Lemma 2.2, the embedding $B^{3/2}_{2,1} \hookrightarrow L^\infty$ and (4.7), we obtain that for

$t \in [0, T]$,

$$\begin{aligned}
&\|u_{k,n}(t)\|_{B^{5/2}_{2,\infty}} - \|u_{k,n}(0)\|_{B^{5/2}_{2,\infty}} \\
&\quad \le C\int_0^t \|F(u_{k,n})\|_{B^{5/2}_{2,\infty}}\,\mathrm{d}\tau + C\int_0^t \|\partial_x u_{k,n}\|_{L^\infty}\|u_{k,n}\|_{B^{5/2}_{2,\infty}}\,\mathrm{d}\tau \\
&\quad \lesssim \int_0^t \|u_{k,n}\|_{B^{3/2}_{2,1}}\|u_{k,n}\|_{B^{5/2}_{2,\infty}}\,\mathrm{d}\tau \lesssim \int_0^t \|u_{k,n}\|_{B^{5/2}_{2,\infty}}\,\mathrm{d}\tau.
\end{aligned}$$

From the above inequality and Lemma 2.9, we have

$$\|u_{k,n}(t)\|_{B^{5/2}_{2,\infty}} \le \|u_{k,n}(0)\|_{B^{5/2}_{2,\infty}}(1 + C_T\ \mathrm{e}^{C_T}) \lesssim n. \tag{4.11}$$

Thanks to (4.11), we arrive at

$$\|v(t)\|_{B^{5/2}_{2,\infty}} \le \|u^{k,n}(t)\|_{B^{5/2}_{2,\infty}} + \|u_{k,n}(t)\|_{B^{5/2}_{2,\infty}} \lesssim n.$$

Combining these results gives rise to the desired estimates. □

4.3 Proof for Theorem 1.3

We firstly rechoose $\widehat{T} \in [0, T]$ such that $\exp\left\{-C\widehat{T}\right\} > \frac{4}{5}$. Then from (4.10), we have

$$\|v\|_{B^{1/2}_{2,\infty}} \lesssim n^{-\frac{3}{2}\exp\left\{-C\widehat{T}\right\}} \lesssim n^{-\frac{6}{5}}, \quad n \gg 1.$$

We will show that for $t \in [0, \widehat{T}]$, $u_{-1,n}$ and $u_{1,n}$ are two sequences which satisfy (1.17)–(1.19).

From (4.7), we see that for $k = -1, 1$, $\|u_{k,n}(t)\|_{B^{3/2}_{2,1}} \lesssim \|u^{k,n}(0)\|_{B^{3/2}_{2,1}}$, which implies (1.17). And (1.18) is given by

$$\|u_{-1,n}(0) - u_{1,n}(0)\|_{B^{3/2}_{2,1}} = \|u^{-1,n}(0) - u^{1,n}(0)\|_{B^{3/2}_{2,1}} \lesssim n^{-1} \xrightarrow[n\to\infty]{} 0. \tag{4.12}$$

For (1.19), we consider

$$\begin{aligned}
\liminf_{n\to\infty}\|u_{-1,n}(t) - u_{1,n}(t)\|_{B^{3/2}_{2,1}} \ge &\liminf_{n\to\infty}\|u^{-1,n}(t) - u^{1,n}(t)\|_{B^{3/2}_{2,1}} \\
&- \lim_{n\to\infty}\|u^{-1,n}(t) - u_{-1,n}(t)\|_{B^{3/2}_{2,1}} \\
&- \lim_{n\to\infty}\|u^{1,n}(t) - u_{1,n}(t)\|_{B^{3/2}_{2,1}}.
\end{aligned} \tag{4.13}$$

By Lemma 4.2 and interpolation inequality, we obtain

$$\|v\|_{B^{3/2}_{2,1}} \le \|v\|^{\frac{1}{2}}_{B^{1/2}_{2,\infty}}\|v\|^{\frac{1}{2}}_{B^{5/2}_{2,\infty}} \lesssim n^{-\frac{3}{5}}n^{\frac{1}{2}} = n^{-\frac{1}{10}},$$

and therefore we deduce that

$$\lim_{n\to\infty}\|u^{-1,n}(t) - u_{-1,n}(t)\|_{B^{3/2}_{2,1}} = \lim_{n\to\infty}\|u^{1,n}(t) - u_{1,n}(t)\|_{B^{3/2}_{2,1}} = 0.$$

Therefore, we have that for $t \in [0, \widehat{T}]$,

$$\begin{aligned}
\liminf_{n\to\infty}\|u_{-1,n}(t) - u_{1,n}(t)\|_{B^{3/2}_{2,1}} &\gtrsim \liminf_{k\to\infty}\|u^{-1,n}(t) - u^{1,n}(t)\|_{B^{3/2}_{2,1}} \\
&\gtrsim \liminf_{n\to\infty}\left(\|n^{-3/2}\sin(nx)\sin t\|_{B^{3/2}_{2,1}} - \|n^{-1}\|_{B^{3/2}_{2,1}}\right) \\
&= \liminf_{n\to\infty}\left(n^{-3/2}\|\sin(nx)\|_{B^{3/2}_{2,1}}|\sin t|\right) \\
&\gtrsim |\sin(t)|.
\end{aligned}$$

Therefore, the solution map defined by (1.13) is indeed not uniformly continuous.

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