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# Action of a Polynomial Matrix on a Vector of Power Series

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***Author's contribution***

*The sole author designed, analyzed and interpreted and prepared the manuscript.*

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## Abstract

The adjoint of the right multiplication of a row vector by a fixed polynomial matrix gives a left operation of the polynomial matrix on column vectors of power series. This explain the polynomial matrix and vector of powers series "multiplication", used to define discrete linear dynamical systems, according to Willems and Oberst theory.

*Keywords: Discrete linear dynamical system; behavior; polynomial operator in the shift; categories and functors; adjoint of a linear mapping; polynomial matrix.*

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## 1 Introduction

In [1], we have explained the polynomial operator in the shifts as the adjoint of the linear mapping defined on the vector space of polynomials, which is the *multiplication* by a fixed polynomial. In

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this article, we are going to generalize this result by showing that the adjoint of the linear mapping defined on rows vectors of polynomials, defined by the right multiplication by a fixed polynomial matrix, defines an operation on column vectors of powers series. This gives deeper explanation and interpretation of this matrix operation than we gave in [1]. Recall that this matrix operation, though fundamental in defining discrete linear dynamical systems, remains mathematically unexplained or uninterpreted by the other authors, see [2, 3, 4, 5, 7, 6, 8, 9, 10, 11, 12, 13, 14].

# 2 Basic Data

## 2.1 Notations

We recall the notations we used in [1] . For a commutative field $\mathbb{F}$ and an integer $r \geqslant 1$, let $\mathbb{F}^{\mathbb{N}^r}$ be the vector space of the sequences of elements of $\mathbb{F}$ indexed by $\mathbb{N}^r$ :

$$\mathbb{F}^{\mathbb{N}^r} = \{W : \mathbb{N}^r \longrightarrow \mathbb{F},\ W \longmapsto W(\alpha) = W_\alpha\}$$

and $\mathbb{F}^{(\mathbb{N}^r)}$ the $\mathbb{F}$-subspace of $\mathbb{F}^{\mathbb{N}^r}$ consisting of those of finite support :

$$\mathbb{F}^{\mathbb{N}^r} = \{W \in \mathbb{F}^{\mathbb{N}^r} \mid \mathrm{Supp}(W) \text{ is finite}\},$$

where $\mathrm{Supp}(W) = \{\alpha \in \mathbb{N}^r \mid W_\alpha \neq 0\}$. Let be $X_1, \ldots, X_r$ (resp. $Y_1, \ldots, Y_r$) be *variables*. The letter $X$ (resp. $Y$) will denote $X_1, \ldots, X_r$ (resp $Y_1, \ldots, Y_r$) and for $\alpha \in \mathbb{N}^r$ we define $X^\alpha$ (resp. $Y^\alpha$) by

$$X^\alpha = X_1^{\alpha_1} \cdots X_r^{\alpha_r} \quad (\text{resp. } Y^\alpha = Y_1^{\alpha_1} \cdots Y_r^{\alpha_r}).$$

For $\alpha \in \mathbb{N}^r$, let $\delta_\alpha$ be the mapping

$$\begin{aligned} \delta_\alpha : \mathbb{N}^r &\longrightarrow \mathbb{F} \\ \beta &\longmapsto \delta_\alpha(\beta) = \begin{cases} 0, & \text{if} \quad \alpha \neq \beta, \\ 1, & \text{if} \quad \alpha = \beta. \end{cases} \end{aligned} \tag{2.1}$$

Then $\delta_\alpha \in \mathbb{F}^{(\mathbb{N}^r)}$ with $\mathrm{Supp}(\delta_\alpha) = \{\alpha\}$.

Let $\mathbf{D} = \mathbb{F}[X_1, \ldots, X_r] = \mathbb{F}[X]$ be the $\mathbb{F}$-vector space of the polynomials with the $r$ variables $X_1, \ldots, X_r$ and $\mathbf{A} = \mathbb{F}[[Y_1, \ldots, Y_r]] = \mathbb{F}[[Y]]$ that of the formal power series with the $r$ variables $Y_1, \ldots, Y_r$. The family $(X^\alpha)_{\alpha \in \mathbb{N}^r}$ is an $\mathbb{F}$-base of $\mathbf{D}$, thus an element of $\mathbf{D}$ can be written uniquely as

$$d(X) = \sum_{\alpha \in \mathbb{N}^r} d_\alpha X^\alpha \quad \text{with} \quad d_\alpha \in \mathbb{F} \quad \text{for all } \alpha \in \mathbb{N}^r,$$

where $d_\alpha = 0$ except for a finite number of $\alpha$'s. An element $W(Y)$ of $\mathbf{A}$ can be uniquely expressed as

$$W(Y) = \sum_{\alpha \in \mathbb{N}^r} W_\alpha Y^\alpha \quad \text{with} \quad W_\alpha \in \mathbb{F} \quad \text{for all} \quad \alpha \in \mathbb{N}^r.$$

Therefore, we get the $\mathbb{F}$-vector spaces isomorphisms

$$\begin{aligned} \mathbf{D} = \mathbb{F}[X_1, \ldots, X_r] &\cong \mathbb{F}^{(\mathbb{N}^r)} \\ X^\alpha &\longleftrightarrow \delta_\alpha \quad \text{(and then extending this by linearity)} \end{aligned}$$

and

$$\begin{aligned} \mathbf{A} = \mathbb{F}[[Y_1, \ldots, Y_r]] &\cong \mathbb{F}^{\mathbb{N}^r} \\ W(Y) = \sum_{\alpha \in \mathbb{N}^r} W_\alpha Y^\alpha &\longleftrightarrow W = (W_\alpha)_{\alpha \in \mathbb{N}^r}. \end{aligned}$$

By these isomorphisms, we may identify $X^\alpha$ (resp. $Y^\alpha$) with the element $\delta_\alpha$ of $\mathbb{F}^{(\mathbb{N}^r)}$ (resp. of $\mathbb{F}^{\mathbb{N}^r}$). If $W \in \mathbb{F}^{\mathbb{N}^r}$, we may write $W = (W_\alpha)_{\alpha\in\mathbb{N}^r}$, where $W_\alpha = W(\alpha)$ for all $\alpha \in \mathbb{N}^r$. Finally, we may write the following identifications

$$W = (W_\alpha)_{\alpha\in\mathbb{N}^r} = \sum_{\alpha\in\mathbb{N}^r} W_\alpha Y^\alpha = W(Y). \tag{2.2}$$

The set $\mathbb{F}^{\mathbb{N}^r}$ (resp. $\mathbb{F}^{(\mathbb{N}^r)}$) is also denoted by $\mathbf{A}$ (resp. $\mathbf{D}$). Let $h \geqslant 1$ be an integer. The cartesian product $\mathbf{A} \times \cdots \times \mathbf{A}$ (resp. $\mathbf{D} \times \ldots \times \mathbf{D}$) ($h$ times) is denoted by $\mathbf{A}^h$ (resp. $\mathbf{D}^h$). We see $\mathbf{A}^h$ as a set of column vectors and $\mathbf{D}^h$ as a set of rows vectors. The set

$$\begin{aligned} B_h = \{X^\rho e_j^{(h)} = X_1^{\rho_1} \cdots X_r^{\rho_r} e_j^{(h)} \mid \rho = (\rho_1, \ldots, \rho_r) \in \mathbb{N}^r \text{ and} \\ e_j^{(h)} = \underbrace{(0, \ldots, 1, \ldots, 0)}_{1 \text{ at the } j\text{-th position}} \in \mathbf{D}^h \text{ for } j = 1, \ldots, h\} \end{aligned} \tag{2.3}$$

is an $\mathbb{F}$-basis of $\mathbf{D}^h$. Indeed, an element of $\mathbf{D}^h$ is of the form

$$d(X) = (d_1(X), \ldots, d_j(X), \ldots, d_h(X)) \tag{2.4}$$

where

$$d_j(X) = \sum_{\rho\in\mathbb{N}^r} d_{j\rho} X^\rho \in \mathbf{D} \text{ for } j = 1, \ldots, h, \text{ the sum being finite.} \tag{2.5}$$

Writing $d(X)$ in the following form,

$$\begin{aligned} d(X) &= (d_1(X), 0, \ldots, 0) + \cdots + \underbrace{(0, \ldots, d_j(X), \ldots, 0)}_{d_j(X) \text{ at the } j\text{-th position}} + \cdots + (0, \ldots, d_h(X)) \\ &= \sum_{j=1}^{h} d_j(X) e_j^{(h)} \end{aligned}$$

and using the expression of $d_j(X)$ in (2.5), we have

$$\begin{aligned} d(X) &= \sum_{j=1}^{h} (\sum_{\rho\in\mathbb{N}^r} d_{j\rho} X^\rho) e_j^{(h)} \\ &= \sum_{1\leqslant j\leqslant h, \rho\in\mathbb{N}^r} d_{j\rho} X^\rho e_j^{(h)}, \end{aligned} \tag{2.6}$$

so that $B_h$ generates $\mathbf{D}^l$ as an $\mathbb{F}$-vector space. Now, suppose that we have

$$\sum_{1\leqslant j\leqslant h, \rho\in\mathbb{N}^r} d_{j\rho} X^\rho e_j^{(h)} = 0, \tag{2.7}$$

where $d_{j\rho} \in \mathbb{F}$ for $j = 1, \ldots, h$ and $\rho \in \mathbb{N}^r$, with $d_{j\rho} = 0$ except for a finite number of $\rho'$s. This assures that the sum (2.7) is finite. We then construct the polynomials

$$d_j(X) = \sum_{\rho\in\mathbb{N}^r} d_{j\rho} X^\rho \in \mathbf{D},$$

and the polynomial vector

$$d(X) = (d_1(X), \ldots, d_j(X), \ldots, d_h(X)) \in \mathbf{D}^l.$$

We are in the situation of the equation (2.4). Using (2.6) and (2.7), we get

$$d(X) = (d_1(X), \ldots, d_j(X), \ldots, d_h(X)) = 0,$$

hence $d_j(X) = 0$ for $j = 1, \ldots, h$.This ensures that $d_{j\rho} = 0$ for $j = 1, \ldots, h$ and $\rho \in \mathbb{N}^r$. Coming back to (2.7), we conclude that the elements $X^\rho e_j^{(h)}$ of $B_h$ are linearly independent. We have then proven that $B_h$ is an $\mathbb{F}$-basis of $\mathbf{D}^h$.

For integers $k, l \geqslant 1$, the set of matrices with $k$ rows and $l$ columns with coefficients in $\mathbf{A}$ (resp. in $\mathbf{D}$) is denoted $\mathbf{A}^{k,l}$ (resp. $\mathbf{D}^{k,l}$). According to our previous notation, $\mathbf{A}^k$ (resp. $\mathbf{D}^l$) denotes $\mathbf{A}^{k,1}$ (resp. $\mathbf{D}^{1,l}$). An element $R(X) \in \mathbf{D}^{k,l}$ is of the form

$$R(X) = (R_{ij}(X))_{1 \leqslant i \leqslant k, 1 \leqslant j \leqslant\ l}$$

where $R_{ij}(X) \in \mathbf{D}$ for $i = 1, \ldots, k$ and $j = 1, \ldots, l$.

Let Vect($\mathbb{F}$) be the category of the vector spaces over $\mathbb{F}$. For $E, F \in$ Vect($\mathbb{F}$), the set of morphisms from $E$ into $F$ is $\mathbf{Hom}_\mathbb{F}(E, F)$, the set of linear mappings from $E$ to $F$. We will use the functor

$$\begin{aligned} \mathrm{Hom}_\mathbb{F}(-, \mathbb{F}) : \mathrm{Vect}(\mathbb{F}) &\longrightarrow \mathrm{Vect}(\mathbb{F}) \\ E &\longmapsto \mathrm{Hom}_\mathbb{F}(E, \mathbb{F}) \\ (f : E \longrightarrow F) &\longmapsto \left\{ \begin{aligned} \mathrm{Hom}_\mathbb{F}(f, \mathbb{F}) : \mathrm{Hom}_\mathbb{F}(F, \quad \mathbb{F}) &\longrightarrow \mathrm{Hom}_\mathbb{F}(E, \mathbb{F}) \\ u &\longmapsto u \circ f. \end{aligned} \right. \end{aligned} \tag{2.8}$$

**Definition 2.1.** Let $E, F \in$ Vect($\mathbb{F}$). The *(functorial) adjoint* or *transpose* of $f$ if is the linear mapping

$$\begin{aligned} \mathbf{Hom}_\mathbb{F}(f, \mathbb{F}) : \mathbf{Hom}_\mathbb{F}(F, \mathbb{F}) &\longrightarrow \mathbf{Hom}_\mathbb{F}(E, \mathbb{F}) \\ u &\longmapsto u \circ f. \end{aligned} \tag{2.9}$$

In [1], theorem 3.3, we proved that given a polynomial $d(X) = \sum_{\beta \in \mathbb{N}^r} d_\beta X^\beta \in \mathbf{D}$, the functorial adjoint of the polynomial multiplication

$$\begin{aligned} d(X) : \mathbf{D} &\longrightarrow \mathbf{D} \\ c(X) &\longmapsto c(X)d(X), \end{aligned}$$

is the *polynomial operation in the shifts*

$$\begin{aligned} d(X) : \mathbf{A} &\longrightarrow \mathbf{A} \\ W(Y) &\longmapsto dX) \circ W(Y) = \sum_{\alpha \in \mathbb{N}^r} (\sum_{\beta \in \mathbb{N}^r} d_\beta W_{\alpha+\beta}) Y^\alpha. \end{aligned} \tag{2.10}$$

We have called the symbol "$\circ$" the "multiplication" of a vector of power series by a polynomial matrix. Using this notation, given a polynomial matrix $R(X) = (R_{ij}(X))_{1 \leqslant i \leqslant k, 1 \leqslant j \leqslant\ l}$ of $\mathbf{D}^{k,l}$, the action of $R(X)$ on a column vector of power series $W(Y) = (W_1(Y), \ldots, W_l(Y))^T \in \mathbf{A}^l$ (where $T$ is the transposition) is usually defined as

$$R(X) \circ W(Y) = \begin{pmatrix} R_1(X) \circ W(Y) \\ \vdots \\ R_k(X) \circ W(Y) \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^l R_{1j}(X) \circ W_j(Y) \\ \vdots \\ \sum_j^l R_{kj}(X) \circ W_j(Y) \end{pmatrix} \in \mathbf{A}^k. \tag{2.11}$$

## 2.2 The problem and the method

According to our notations, we will prove that, once $R(X) \in \mathbf{D}^{k,l}$ is fixed, the adjoint of the linear mapping

$$\begin{aligned} R(X)^T : \mathbf{D}^k &\longrightarrow \mathbf{D}^l \\ c(X) &\longmapsto c(X) \cdot R(X) \end{aligned} \tag{2.12}$$

is the linear mapping defined by

$$\begin{aligned} R(X) : \mathbf{A}^l &\longrightarrow \mathbf{A}^k \\ W(Y) &\longmapsto R(X) \circ W(Y). \end{aligned} \tag{2.13}$$

This will explains the operation " $\circ$ ". Moreover it is a linear mapping of $\mathbf{D}$-modules. In other terms, $\mathbf{Hom}_{\mathbb{F}}(R(X)^T, \mathbb{F}) = R(X)$ (here $R(X)^T$ and $R(X)$ are view as the mappings in (2.12) and (2.13)). We therefore have resolved one of the problems we stated in the conclusion of [1]. It is very interesting that simply taking the adjoint of (2.12) leads to an action of the polynomial $R(X)$ on the elements of $\mathbf{A}^l$, which are vectors of powers series.

For this purpose, we use lemma 3.1 in order to consider an element $W \in \mathbf{A}^l$ as a linear mapping from $\mathbf{D}^l$ to $\mathbb{F}$. Then, starting from the definition of the adjoint of $R(X)^T$, which is the linear mapping $\mathbf{Hom}_{\mathbb{F}}(R(X)^T, \mathbb{F}) = W \circ R(X)^T$, we directly calculate the images under $W \circ R(X)^T$ of the elements of the $\mathbb{F}$-basis $\{X^\rho e_j^{(l)} \mid \rho \in \mathbb{N}^r, j = 1, \dots, h\}$ of $\mathbf{D}^l$. Finally, we simply identify $W \circ R(X)^T$ by these images arranged in a specific matrix form.

# 3 Solution of the Problem

## 3.1 Preliminary results

We need the following lemma ([4], p.60) for our main theorem. Our proof is simpler and more direct.

**Lemma 3.1.** *Let $h \geqslant 1$ be an integer. Then the linear mapping*

$$\mathbf{Hom}_{\mathbb{F}}(\mathbf{D}^h, \mathbb{F}) \longrightarrow \mathbf{A}^h$$

$$f \longrightarrow \begin{pmatrix} (f(X^\rho e_1^{(h)}))_{\rho \in \mathbb{N}^r} \\ \vdots \\ f(X^\rho e_j^{(h)}))_{\rho \in \mathbb{N}^r} \\ \vdots \\ f(X^\rho e_h^{(h)}))_{\rho \in \mathbb{N}^r} \end{pmatrix} = \begin{pmatrix} \sum_{\rho \in \mathbb{N}^r} f(X^\rho e_1^{(h)}) Y^\rho \\ \vdots \\ \sum_{\rho \in \mathbb{N}^r} f(X^\rho e_j^{(h)}) Y^\rho \\ \vdots \\ \sum_{\rho \in \mathbb{N}^r} f(X^\rho e_h^{(h)}) Y^\rho \end{pmatrix} \tag{3.1}$$

*is an isomorphism of vector spaces. Therefore, we may write*

$$\mathbf{Hom}_{\mathbb{F}}(\mathbf{D}^h, \mathbb{F}) = \mathbf{A}^h\,. \tag{3.2}$$

*Proof.* An element $f \in \mathbf{Hom}_{\mathbb{F}}(\mathbf{D}^h, \mathbb{F})$ is uniquely defined by the images $(f(X^\rho e_j^{(h)}))_{\rho \in \mathbb{N}^r, j=1,\dots,h}$ of the $\mathbb{F}$-basis $B_h = \{X^\rho e_j^{(h)} \mid \rho \in \mathbb{N}^r, j = 1, \dots, h\}$ of $\mathbf{D}^h$, which may be arbitrary elements of $\mathbb{F}$. We may arrange these images into the form of the first matrix in (3.1). By (2.2), we may write

$$(f(X^\rho e_j^{(h)}))_{\rho \in \mathbb{N}^r} = \sum_{\rho \in \mathbb{N}^r} f(X^\rho e_j^{(h)}) Y^\rho \in \mathbf{A}^h \text{ for } j = 1, \dots, h.$$

Thus the two matrices in (3.1) are equal.

## 3.2 Polynomial matrix and vector of power series multiplication

Here is the main result :

**Theorem 3.2.** *Let $R(X) \in \mathbf{D}^{k,l}$. The adjoint of the $\mathbb{F}$-linear mapping*

$$\begin{aligned} R(X)^T : \mathbf{D}^k &\longrightarrow \mathbf{D}^l \\ c(X) &\longmapsto c(X) \cdot R(X) \end{aligned} \tag{3.3}$$

*is the* **D***-linear mapping*

$$\begin{aligned} R(X) : \mathbf{A}^l &\longrightarrow \mathbf{A}^k \\ W(Y) &\longmapsto R(X) \circ W(Y) \end{aligned} \tag{3.4}$$

*where* $R(X) \circ W(Y)$ *is defined by* (2.11).

*Proof.* Applying the functor $\mathbf{Hom}_{\mathbb{F}}(-, \mathbb{F})$ to (3.3), we get

$$\begin{aligned} \mathbf{Hom}_{\mathbb{F}}(R(X)^T, \mathbb{F}) : \mathbf{Hom}_{\mathbb{F}}(\mathbf{D}^l, \mathbb{F}) &\longrightarrow \mathbf{Hom}_{\mathbb{F}}(\mathbf{D}^k, \mathbb{F}) \\ f &\longmapsto \mathbf{Hom}_{\mathbb{F}}(R(X)^T, \mathbb{F})(f) = f \circ R(X)^T. \end{aligned}$$

By (3.2), and considering $f \in \mathbf{Hom}_{\mathbb{F}}(\mathbf{D}^l, \mathbb{F}) = \mathbf{A}^l$ as an element $W \in \mathbf{A}^l$, we have

$$\begin{aligned} \mathbf{Hom}_{\mathbb{F}}(R(X)^T, \mathbb{F}) : \mathbf{A}^l &\longrightarrow \mathbf{A}^k \\ W &\longmapsto \mathbf{Hom}_{\mathbb{F}}(R(X)^T, \mathbb{F})(W) = W \circ R(X)^T. \end{aligned} \tag{3.5}$$

In the expression $W \circ R(X)^T$ of (3.5), it is useful to consider $W$ again as a linear form from $\mathbf{D}^l$ to $\mathbb{F}$ (acoding to (3.2)) . The symbol $\circ$ is the composition of mappings. Now, we are going to find the image under $W$ of a polynomial vector $d(X) \in \mathbf{D}^l$. Set $d(X) = (d_1(X), \ldots, d_j(X), \ldots, d_l(X))$. Each $d_j(X)$ is of the form

$$d_j(X) = \sum_{\rho \in \mathbb{N}^r} d_{j\rho} X^\rho,$$

where the sequence $(d_{j\rho})_{\rho \in \mathbb{N}^r}$ of elements of $\mathbb{F}$ is with finite support for $j = 1, \ldots, , l$. Using the $\mathbb{F}$-basis $B_l = \{X^\rho e_j^{(l)} \mid \rho \in \mathbb{N}^r, j = 1 \ldots l\}$ of $\mathbf{D}^l$, we obtain, by (2.6)

$$d(X) = \sum_{j=1}^{l} \sum_{\rho \in \mathbb{N}^r} d_{j\rho} X^\rho e_j^{(l)}.$$

Taking the image of $d(X)$ by $W$, we get

$$W(d(X)) = W(\sum_{j=1}^{l} \sum_{\rho \in \mathbb{N}^r} d_{j\rho} X^\rho e_j^{(l)}) = \sum_{j=1}^{l} \sum_{\rho \in \mathbb{N}^r} d_{j\rho} W(X^\rho e_j^{(l)}).$$

Now, write $W(X^\rho e_j^{(l)}) = W_{j\rho} \in \mathbb{F}$; we then have

$$W(d(X)) = \sum_{j=1}^{l} \sum_{\rho \in \mathbb{N}^r} d_{j\rho} W_{j\rho}. \tag{3.6}$$

Now, we are ready to calculate $(W \circ R(X)^T)(X^\rho e_i^{(k)})$, the images of the polynomial vectors $X^\rho e_i^{(k)}$ of the elements of the $\mathbb{F}$-basis $\{X^\rho e_j^{(h)} \mid \rho \in \mathbb{N}^r, j = 1, \ldots, h\}$ of $\mathbf{D}^k$ by the linear form $W \circ R(X)^T$. Let $R(X) = (R_{ij}(X))_{i=1,\ldots,k,j=1,\ldots l}$ and $R_{ij}(X) = \sum_{\alpha \in \mathbb{N}^r} R_{ij\alpha} X^\alpha$, where the sequence $(R_{ij\alpha})_{\alpha \in \mathbb{N}^r}$ is with finite support for $i = 1, \ldots, k$ and $j = 1, \ldots, l$. For simplicity, we will write $R^T$ for the linear mapping $R(X)^T$, therefore $W \circ R^T$ instead of $W \circ R(X)^T$. We have

$$\begin{aligned} R^T(X^\rho e_i^{(k)}) = X^\rho e_i^{(k)} \cdot R(X) &= \underbrace{(0, \ldots, X^\rho, \ldots, 0)}_{X^\rho \text{ at the } i\text{-th position}} \cdot \begin{pmatrix} R_{11} & \ldots & R_{1j} & \ldots & R_{1l} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ R_{i1} & \ldots & R_{ij} & \ldots & R_{il} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ R_{k1} & \ldots & R_{kj} & \ldots & R_{kl} \end{pmatrix} \\ &= (X^\rho R_{i1}, \ldots, X^\rho R_{ij}, \ldots, X^\rho R_{il}) \\ &= (\sum_{\alpha \in \mathbb{N}^r} R_{i1\alpha} X^{\alpha+\rho}, \ldots, \sum_{\alpha \in \mathbb{N}^r} R_{ij\alpha} X^{\alpha+\rho}, \ldots, \sum_{\alpha \in \mathbb{N}^r} R_{il\alpha} X^{\alpha+\rho}) \end{aligned} \tag{3.7}$$

Taking the image of $R^T(X^\rho e_i^{(k)})$ by $W$, we have, by (3.6), where $d(X)$ is replaced by the expression of $R^T(X^\rho e_i^{(k)})$ in the last equation of (3.7) ,

$$(W \circ R^T)(X^\rho e_i^{(k)}) = W(R^T(X^\rho e_i^{(k)})) = \sum_{j=1}^{l} \sum_{\alpha \in \mathbb{N}^r} R_{ij\alpha} W_{j(\alpha+\rho)}. \tag{3.8}$$

Thus, the images of the elements of the base $B$ under $W \circ R^T$ are given by the following vector of power series

$$\begin{pmatrix} \sum_{\rho \in \mathbb{N}^r} (\sum_{j=1}^{l} \sum_{\alpha \in \mathbb{N}^r} R_{1j\alpha} W_{j(\alpha+\rho)}) Y^\rho \\ \vdots \\ \sum_{\rho \in \mathbb{N}^r} (\sum_{j=1}^{l} \sum_{\alpha \in \mathbb{N}^r} R_{ij\alpha} W_{j(\alpha+\rho)}) Y^\rho \\ \vdots \\ \sum_{\rho \in \mathbb{N}^r} (\sum_{j=1}^{l} \sum_{\alpha \in \mathbb{N}^r} R_{kj\alpha} W_{j(\alpha+\rho)}) Y^\rho \end{pmatrix}, \tag{3.9}$$

where the coefficients of the power series in the $i$-th row represent the image of $X^\rho e_i^{(k)}$ for $\rho \in \mathbb{N}^r$ (see lemma 3.1). We can rearrange the $i$-th row of (3.9) into the following form,

$$\sum_{\rho \in \mathbb{N}^r} (\sum_{j=1}^{l} \sum_{\alpha \in \mathbb{N}^r} R_{ij\alpha} W_{j(\alpha+\rho)}) Y^\rho = \sum_{j=1}^{l} (\sum_{\rho \in \mathbb{N}^r} (\sum_{\alpha \in \mathbb{N}^r} R_{ij\alpha} W_{j(\alpha+\rho)}) Y^\rho),$$

and comparing with (2.10), we have

$$\sum_{j=1}^{l} (\sum_{\rho \in \mathbb{N}^r} (\sum_{\alpha \in \mathbb{N}^r} R_{ij\alpha} W_{j(\alpha+\rho)}) Y^\rho) = \sum_{j=1}^{l} R_{ij}(X) \circ W_j(Y). \tag{3.10}$$

Using this, the matrix (3.9) finally equals to the following matrix

$$\begin{pmatrix} \sum_{j=1}^{l} R_{1j}(X) \circ W_j(Y) \\ \vdots \\ \sum_{j}^{l} R_{kj}(X) \circ W_j(Y) \end{pmatrix},$$

which is the same as (2.11) and $W \circ R(X)^T$ may be identified with these matrices. For the proof of the **D**-linearity, see [4]. This completes the proof of the theorem.

The matrix $W \circ R(X)^T$ being an element of $\mathbf{A}^k$, constructed from the matrix $R(X) \in \mathbf{D}^{k,l}$ and the vector $W(Y) \in \mathbf{A}^l$, it can be considerated a result of an operation of $R(X)$ on $W(Y)$, which is explains the notation "$R(X) \circ W(Y)$".

# 4 Conclusions

As we have seen, taking the adjoint of a simple polynomial vector and polynomial matrix multiplication leads to an amazing and unexpected result. It possible to multiply a vector of power series by a polynomial matrix. Apparently, these two objets have nothing in common. This illustrate the utility of the correspondence between set of polynomials and power series.

# Competing Interests

Author has declared that no competing interests exist.

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